Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section, Round 2 solutions)

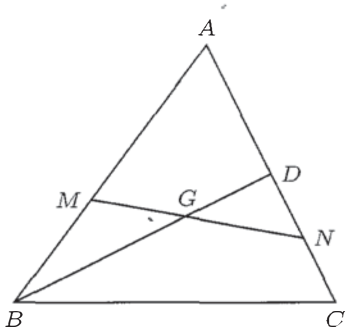
1. Let D be the midpoint of AC. Since $\frac{CN}{NA} < 1$, N lies in the segment CD. Let G be the intersection of BD and MN. By Menelaus' Theorem applied to the line MN and triangle ABD,

$$\frac{DG}{GB} \cdot \frac{BM}{MA} \cdot \frac{AN}{ND} = 1.$$

Thus

$$\begin{aligned}\frac{BG}{GD} &= \frac{BM}{MA} \cdot \frac{AN}{ND} = \left(1 - \frac{CN}{NA}\right) \cdot \frac{AN}{ND} \\ &= \frac{NA - CN}{ND} = \frac{(2CD - CN) - CN}{ND} \\ &= \frac{2ND}{ND} = 2.\end{aligned}$$

Therefore, G is the centroid of ABC.



2. We have $n^2 \equiv 1 \pmod 3$. Thus $n = 3k + 1$ or $3k + 2$ for some nonnegative integer k.

(i) $n = 3k + 1$. After simplifying, we have $2^m = 3k^2 + 2k = k(3k + 2)$. Thus k and $3k + 2$ are both powers of 2. It is clear that $k = 2$ is a solution and $k = 1$ is not. If $k = 2^p$, where $p \geq 2$, then $3k + 2 = 2(3 \cdot 2^{p-1} + 1)$ is not a power of 2 as $3 \cdot 2^{p-1} + 1$ is odd. We have one solution: $n = 7, m = 4$.

(ii) $n = 3k + 2$: Again we have $2^m = 3k^2 + 4k + 1 = (3k + 1)(k + 1)$ and both $k + 1$ and $3k + 1$ must be powers of 2. Both $k = 0, 1$ are solutions. When $k = 0$, $m = 0$, which is not admissible. For $k > 1$, we have $3k + 1 = 2k + (k + 1) > 2k + 2$ and therefore $4(k + 1) > 3k + 1 > 2(k + 1)$. Hence if $k + 1 = 2^p$ for some positive integer p, then

$2^{p+2} > 3k+1 > 2^{p+1}$ and we conclude that $3k+1$ cannot be a power of 2. Thus there is one solution in this case: $(n, m) = (5, 3)$.

et A be an n-element subset of $\{1, 2, \ldots, 2009\}$ with the property that the difference between any two numbers in A is not a prime number. Find the largest possible value of n. Find a set with this number of elements. (Note: 1 is not a prime number.)

3. If $n \in A$, then $n + i \notin A$, $i = 2, 3, 5, 7$. Among $n+1, n+4, n+6$ at most one can be in A. Thus among any 8 consecutive integers, at most 2 can be in S. Hence $|A| \leq 2\lceil 2009/8 \rceil = 504$. Such a set is $\{4k+1 : k = 0, 1, \ldots, 502\}$.

4. We give a proof of the general case. Consider the expansion of

$$(ax_1^2 + bx_1 + c)(ax_2^2 + bx_2 + c) \cdots (ax_n^2 + bx_n + c).$$

The term in $a^i b^j c^k$, where $i + j + k = n$ is

$$a^i b^j c^k [(x_1 x_2 \ldots x_i)^2 (x_{i+1} x_{i+2} \ldots x_{i+j}) + \cdots].$$

There are altogether $\binom{n}{i}\binom{n-i}{j}$ terms in the summation. (We choose i factors from which we take ax_t^2. From the remaining $n - i$ factors, we choose j to take the terms bx_s.) By symmetry, the number of terms containing x_i^2 is a constant, as is the number of terms containing the term x_i. Thus, when the terms in the summation are multiplied together, we get $(x_1 x_2 \ldots x_n)^p = 1$ for some p. (For our purpose, it is not necessary to compute p. In fact $p = 2\binom{n-1}{i-1}\binom{n-i}{j} + \binom{n-1}{j-1}\binom{n-j}{i} = \frac{2i+j}{n}\binom{n}{i}\binom{n-i}{j}$.) By the AM-GM inequality, we have

$$a^i b^j c^k [(x_1 x_2 \ldots x_i)^2 (x_{i+1} x_{i+2} \ldots x_{i+j}) + \cdots] \geq a^i b^j c^k \binom{n}{i}\binom{n-i}{j}.$$

Hence

$$(ax_1^2 + bx_1 + c) \cdots (ax_n^2 + bx_n + c) \geq \sum_{i+j+k=n} a^i b^j c^k \binom{n}{i}\binom{n-i}{j} = (a+b+c)^n = 1.$$

5. The number of arrows that that hit zone 1 is $< 30 \cdot 16/4 = 120$. If contestant i hits zone 1 a_i times, zone 2 b_i times and miss the target c_i times, then the total score is $10a_i + 5b_i = 5a_i + 5(a_i + b_i) = 5a_i + 5(16 - c_i) = 80 + 5(a_i - c_i)$. Suppose the scores are all distinct, then the 30 numbers $a_i - c_i$ must all be distinct. By the pigeonhole principle, half of these 30 numbers are either positive or negative. We consider the "positive" case. Without loss of generality, let $a_i - c_i > 0$ for $i = 1, \ldots, 15$. Then $a_i - c_i \geq i$. Therefore $a_i \geq i$. Hence $a_1 + \cdots + a_{15} \geq 120$. But $a_1 + \cdots + a_{30} < 120$, and we have a contradiction. The "negative" case is similar.