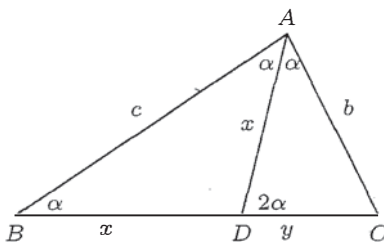


Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Junior Section, Round 2 solutions)

1. Let AD be the angle bisector of $\angle A$. Then $\triangle ABC \simeq \triangle DAC$. Thus $AB/DA = AC/DC = BC/AC$. Let $BD = x$ and $DC = y$. Then $c/x = b/y = a/b$. Thus $cb = ax, b^2 = ay$. Thus $b^2 + cb = ax + ay$ and hence $b(b + c) = a^2$.



2. We solve the general case of $2n$ -digit integers where $n \geq 2$. There are $10^{2n} - 10^{2n-1}$ $2n$ -digit integers. There are $10^n - 10^{n-1}$ n -digit integers. Consider all the products of pairs of n -digit integers. The total number P of such products satisfies

$$\begin{aligned} P &\leq 10^n - 10^{n-1} + \frac{(10^n - 10^{n-1})(10^n - 10^{n-1} - 1)}{2} \\ &= \frac{10^{2n} - 10^{2n-1} - (10^{2n-1} - 10^{2n-2} - 10^n + 10^{n-1})}{2} \\ &< \frac{10^{2n} - 10^{2n-1}}{2}. \end{aligned}$$

These products include all the numbers in M . Thus $|M| < |N|$.

3. Two-digit numbers which contain three distinct prime factors are:

$$30 = 2 \cdot 3 \cdot 5, 42 = 2 \cdot 3 \cdot 7, 60 = 4 \cdot 3 \cdot 5, 66 = 2 \cdot 3 \cdot 11, 70 = 2 \cdot 5 \cdot 7, 78 = 2 \cdot 3 \cdot 13, 84 = 4 \cdot 3 \cdot 7$$

From here, we conclude that $a_i = 6$ for $i = 1, 2, \dots, 2007$ and a_{2008} is either 6 or 0.

4. If x is the smallest integer in S such that $x \equiv i \pmod{3}$, then $x + 3k \in S$ and $x - 3(k + 1) \notin S$ for all $k \geq 0$. We have 3 is the smallest multiple of 3 that is in S ; 50 is smallest number in S that is $\equiv 2 \pmod{3}$ and 100 is the smallest number in S that is $\equiv -1 \pmod{3}$. Thus the positive numbers not in S are $1, 4, \dots, 97$ and $2, 5, \dots, 47$. Their sum is

$$\frac{33(97 + 1)}{2} + \frac{16(2 + 47)}{2} = 2009.$$

5. The left hand side is

$$\begin{aligned} & a^5 x_1 x_2 \dots x_5 + a^4 b (x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + \dots + x_2 x_3 x_4 x_5) \\ & + a^3 b^2 (x_1 x_2 x_3 + x_1 x_2 x_4 + \dots + x_3 x_4 x_5) \\ & + a^2 b^3 (x_1 x_2 + x_1 x_3 + \dots + x_4 x_5) + a b^4 (x_1 + x_2 + \dots + x_5) + b^5 \\ & \geq a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5 = (a + b)^5 = 1. \end{aligned}$$

The last is true since by AM-GM inequality,

$$\begin{aligned} x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + \dots + x_2 x_3 x_4 x_5 & \geq 5(x_1 x_2 x_3 x_4 x_5)^{4/5} = 5 \\ x_1 x_2 x_3 + x_1 x_2 x_4 + \dots + x_3 x_4 x_5 & \geq 10(x_1 x_2 x_3 x_4 x_5)^{6/10} = 10 \\ x_1 x_2 + x_1 x_3 + \dots + x_4 x_5 & \geq 10(x_1 x_2 x_3 x_4 x_5)^{4/10} = 10 \\ x_1 + x_2 + \dots + x_5 & \geq 5(x_1 x_2 x_3 x_4 x_5)^{1/5} = 5 \end{aligned}$$

(Note. For a proof of the general case, see Senior Q4)