**SMO Senior 2024 Rd.2**

1. In an acute triangle $ABC$, $AC > AB$, $D$ is the point on $BC$ such that $AD = AB$. Let $\omega_1$ be the circle through $C$ tangent to $AD$ at $D$, and $\omega_2$ the circle through $C$ tangent to $AB$ at $B$. Let $F(\neq C)$ be the second intersection of $\omega_1$ and $\omega_2$. Prove that $F$ lies on $AC$.

2. Find all integer solutions of the equation
$$y^2 + 2y = x^4 + 20x^3 + 104x^2 + 40x + 2003.$$

3. Find the smallest positive integer $n$ for which there exist integers $x_1 < x_2 < \cdots < x_n$ such that every integer from 1000 to 2000 can be written as a sum of some of the integers from $x_1, x_2, \dots, x_n$ without repetition.

4. Suppose $p$ is a prime number and $x, y, z$ are integers satisfying $0 < x < y < z < p$. If $x^3, y^3, z^3$ have equal remainders when divided by $p$, prove that $x^2 + y^2 + z^2$ is divisible by $x + y + z$.

5. Let $a_1, a_2, \ldots$ be a sequence of positive numbers satisfying, for any positive integers $k, l, m, n$ such that $k + n = m + l$,
$$\frac{a_k + a_n}{1 + a_k a_n} = \frac{a_m + a_l}{1 + a_m a_l}.$$
Show that there exist positive numbers $b, c$ so that $b \leq a_n \leq c$ for any positive integer $n$.