Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Senior Section, Round 2 Solutions)

1. Yes. Multiplying both sides by $(2+xy)(2+yz)(2+zx)$, we get

$$F := (x-y)(2+yz)(2+zx)+(y-z)(2+xy)(2+zx)+(z-x)(2+zx)(2+xy)=0.$$

Now regard F as a polynomial in x. Since $F=0$ when $x=y$, $x-y$ is a factor of F. Similarly, $y-z$ and $z-x$ are also factors of F. Since F is of degree 3,

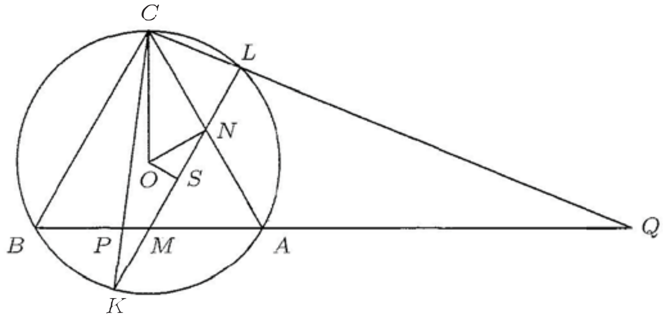
$$F \equiv k(x-y)(y-z)(z-x)$$

for some constant k. By letting $x=1$, $y=-1$, $z=0$, we have $k=2$. Thus

$$F \equiv 2(x-y)(y-z)(z-x)$$

2. See Junior Section Round 2, Question 5.

3. Let the radius of the circumcircle be $R=4t$ and the centre be O. Let S be the foot of the perpendicular from O to MN. Then $OS=ON/2=t$, $KS=\sqrt{16t^2-t^2}=\sqrt{15}t$, $MS=\sqrt{4t^2-t^2}=\sqrt{3}t$. Thus $KM=KS-MS=(\sqrt{15}-\sqrt{3})t$. Since $\triangle PMK \sim \triangle PBC$ and $BC=\sqrt{24}t$, we have $PM/PB=MK/BC=(\sqrt{5}-1)/4$, i.e., $PM=(\sqrt{5}-1)PB/4$. Thus $PA=PM+MA=PM+BM=2PM+PB=(\sqrt{5}+1)PB/2$ and $BA=PA+PB=(\sqrt{5}+3)PB/2$. Therefore $PA^2=PB\cdot BA$. Similarly, $QA^2=QB\cdot BA$. Thus $PA^2\cdot QB=QA^2\cdot PB$.



Second solution: Note that $\angle BCL = \angle CBK$ and $\angle BKC = 60°$. Thus $\angle BCL + \angle BCP = \angle CBK + \angle BCK = 180° - \angle BKC = 120°$. This implies that $\angle BQC = 180° - \angle CBQ - \angle BCQ = 120° - \angle BCL = \angle BCP$, and therefore $\triangle BCQ \sim \triangle BPC$. Thus, $CQ/PC = BQ/BC = BC/BP$, i.e., $\left(\frac{CQ}{PC}\right)^2 = \frac{BQ}{BC} \cdot \frac{BC}{BP} = \frac{BQ}{BP}$. Now since $KL \parallel BC$, A is also midpoint along the arc of KL. Thus, $\angle KCA = \angle ACL$. By the angle bisector theorem on $\triangle PCQ$, we have $CQ/CP = AQ/AP$. Combining, we get $\left(\frac{AQ}{AP}\right)^2 = \left(\frac{CQ}{PC}\right)^2 = \frac{BQ}{BP}$, i.e., $PA^2 \cdot QB = QA^2 \cdot QB$.

4. Suppose on the contrary that this is not possible. We say that a row or a column is *covered* if it contains one of the chosen persons. Choose 32 persons so that the total number of covered rows and columns is maximum. Without loss of generality, assume that column 1 is not covered. Then no pair of twins can be in column 1. The counterparts of these 8 persons must be chosen. By the maximality condition, these 8 persons must be the only chosen persons either in their row or column. If a row contains only 1 chosen person, then there are 6 other persons (excluding those in column 1) in that row who are not chosen. Since 32 persons are not chosen, we have at most 4 such rows. Similarly, there are at most 3 such columns and we have a contradiction because we have 8 such rows or columns.

   2nd solution: We'll show that at most 10 persons are needed. Note that if a set of people cover all the rows and columns, then they will still have the same property if two rows or two columns are interchanged. Thus if the configuration obtained by interchanging some pairs of rows and some pairs of columns can be covered by a set of people, then the same set of people will cover the original configuration. We also let M_i denote the configuration obtained by deleting the first i rows and columns of the original configuration. Let us denote a person by (a, b) if he is in the a-th column b-th row. Choose $(1, 1)$. In M_1, there is a person who is not the counterpart of $(1, 1)$. Without loss of generality, let this be $(2, 2)$. Select $(2, 2)$. In general, if no two of $(1, 1)$, $(2, 2)$, …, (i, i), $i < 6$ are twins, then in M_i there is a person who is not among their counterparts. Let this person be $(i + 1, i + 1)$ and choose this person. In this way, we can choose (i, i), $i = 1, 2, \ldots, 6$. Denote the counterparts of these 6 persons by X and the first 6 persons of the 7th and 8th rows and columns by A_1, A_2, A_3, A_4, respectively. Let $|X \cap A_i| = a_i$. Then $a_1 + a_2 + a_3 + a_4 \leq 6$, Assume that $a_1 \geq a_2 \geq a_3 \geq a_4$. Suppose $a_1 = 6$. Choose $(7, 7)$ and choose a person in A_2 and a person in A_4 who are not twins and are not the counterpart of $(7, 7)$. These 9 people will cover all the row and columns. If $a_1 < 6$, then $a_2 \leq 3$, $a_3 \leq 2$ and $a_4 \leq 1$. Then there exist $p_i \in A_i$ such that $p_i \notin X$, $i = 1, 2, 3, 4$ and not two of p_1, p_2, p_3, p_4 are twins. Choose these 4 persons as well and the 10 chosen people will cover all the rows and columns.

5. We have

$$(x+y)/3 = (\sqrt{2x-1} + \sqrt{y+3/4} + \sqrt{y+3/4})/3 \le \sqrt{(2x+2y+1/2)/3}.$$

and

$$(x+y) = \sqrt{2x-1} + \sqrt{y+3/4} + \sqrt{y+3/4} \ge \sqrt{2x+2y+1/2}.$$

Let $t = x+y$. Then we have

$$\begin{aligned} & 2t^2 - 12t - 3 \le 0 \quad \Rightarrow \quad 3 - \sqrt{21/2} \le t \le 3 + \sqrt{21/2} \\ \text{and} \quad & 2t^2 - 4t - 1 \ge 0 \quad \Rightarrow \quad t \ge 1 + \sqrt{3/2}, \quad t \le 1 - \sqrt{3/2}. \end{aligned}$$

Since $t \ge 0$, the maximum value is $3 + \sqrt{21/2}$ and the minimum value is $1 + \sqrt{3/2}$. Note that the maximum value is attained by the solution of

$$x + y = 3 + \sqrt{21/2} \quad \text{and} \quad 2x - 1 = \sqrt{y + 3/4}$$

while the minimum is attained by the solution of

$$x + y = 1 + \sqrt{3/2} \quad \text{and} \quad y + 3/4 = 0.$$