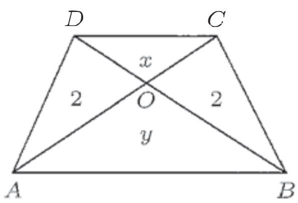
Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Junior Section, Round 2 Solutions)

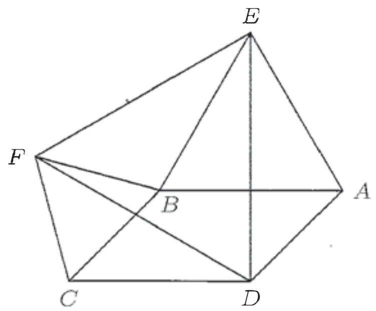
1. Without loss of generality, let $S = 9$. Then $[BOC] = 2$. Since $[ABD] = [ABC]$, we have $[AOD] = [BOC] = 2$. Let $[DOC] = x$ and $[AOB] = y$. Then $x/2 = 2/y$, i.e., $xy = 4$. Also $x + y = 5$. Thus $x(5 - x) = 4$. Solving, we get $x = 1$ and $y = 4$. Since $\triangle DOC \sim \triangle BOA$, we have $x/y = a^2/b^2$. Thus $a/b = 1/2$.



2. We have

$$\begin{aligned}\angle EBF &= 240^\circ - \angle ABC = 240^\circ - (180^\circ - \angle BCD) \\ &= 60^\circ + \angle BCD = \angle DCF\end{aligned}$$

Also $FB = FC$ and $BE = BA = CD$. Thus $\triangle FBE \cong \triangle FCD$. Therefore $FE = FD$. Similarly $\triangle EAD \cong \triangle DCF$. Therefore $ED = DF$. Thus $\triangle DEF$ is equilateral.



3. For $n = 1$ and $n = 2$, the gcd are 1 and 5, respectively. Any common divisor d of $n^2 + 1$ and $(n + 1)^2 + 1$ divides their difference, $2n + 1$. Hence d divides $4(n^2 + 1) - (2n + 1)(2n - 1) = 5$. Thus the possible values are 1 and 5.

4. Let the integers be x and y and assume that $x > y$. First we note that x and y are both nonzero and that their GCD and LCM are both positive by definition. Let M be the GCD, then $|x| = Ma$ and $|y| = Mb$, where a and b are coprime integers. Thus the LCM of x and y is Mab. If $y = 1$, then $M = 1$ and it's easily checked that there is no solution. When $y > 1$, $xy > x + y$. Thus $xy - (x + y) = M + Mab$. After substituting for x and y and simplifying, we have

$$ab(M-1) = 1 + a + b \quad \Rightarrow \quad M = 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} \quad \Rightarrow \quad 1 < M \leq 4.$$

If $M = 2$, then $ab - a - b = 1$, i.e., $(a-1)(b-1) = 2$. Thus $a = 3$, $b = 2$ or $x = 6$, $y = 4$. Similarly, when $M = 3$, we get $2ab - a - b = 1$. Multiplying throughout by 2 and then factorize, we get $(2a-1)(2b-1) = 3$ which gives $x = 6$ and $y = 3$. When $M = 4$, we get $x = y = 4$ which is rejected as x and y are distinct.
Next we consider the case $x > 0 > y$. Then $x = Ma$ and $y = -Mb$. Using similar arguments, we get $x + y - xy = M + Mab$. Thus $M = 1 + \frac{1}{ab} + \frac{1}{a} - \frac{1}{b}$ which yields $1 \leq M \leq 2$. When $M = 1$, we get $a = 1 + b$. Thus the solutions are $b = t$, $a = 1 + t$ or $x = 1 + t$, $y = -t$, where $t \in \mathbb{N}$. When $M = 2$, the equation simplifies to $(a-1)(b+1) = 0$. Thus we get $a = 1$ and b arbitrary as the only solution. The solutions are $x = 2$, $y = -2t$, where $t \in \mathbb{N}$.
Finally, we consider the case $0 > x > y$. Here $x = -Ma$, $y = -Mb$ and $M^2ab + Ma + Mb = M + Mab$. Since $M2ab \geq Mab$ and $Ma + Mb > M$, there is no solution.
Thus the solutions are $(6,3)$, $(6,4)$, $(1+t, -t)$ and $(2, -2t)$ where $t \in \mathbb{N}$.

5. If $f(n) = n + k$, then there are exactly k square numbers less than $f(n)$. Thus $k^2 < f(n) < (k+1)^2$. Now we show that $k = \{\sqrt{n}\}$. We have

$$k^2 + 1 \leq f(n) = n + k \leq (k+1)^2 - 1.$$

Hence

$$\left(k - \frac{1}{2}\right)^2 + \frac{3}{4} = k^2 - k + 1 \leq n \leq k^2 + k = \left(k + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Therefore

$$k - \frac{1}{2} < \sqrt{n} < k + \frac{1}{2},$$

so that $\{\sqrt{n}\} = k$.