Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Senior Section, Round 2)

Saturday, 25 June 2011 **0930-1230**

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1. In the triangle ABC, the altitude at A, the bisector of $\angle B$ and the median at C meet at a common point. Prove that the triangle ABC is equilateral.

2. Determine if there is a set S of 2011 positive integers so that for every pair m, n of distinct elements of S, $|m-n| = (m,n)$. Here (m,n) denotes the greatest common divisor of m and n.

3. Find all positive integers n such that

$$\cos\frac{\pi}{n}\cos\frac{2\pi}{n}\cos\frac{3\pi}{n} = \frac{1}{n+1}.$$

4. Let n and k be positive integers with $n \geq k \geq 2$. For $i = 1, \ldots, n$, let S_i be a nonempty set of consecutive integers such that among any k of them, there are two with nonempty intersection. Prove that there is a set X of $k-1$ integers such that each S_i, $i = 1, \ldots, n$ contains at least one integer in X.

5. Given $x_1, x_2, \cdots, x_n > 0$, $n \geq 5$, show that

$$\frac{x_1x_2}{x_1^2+x_2^2+2x_3x_4} + \frac{x_2x_3}{x_2^2+x_3^2+2x_4x_5} + \cdots + \frac{x_nx_1}{x_n^2+x_1^2+2x_2x_3} \leq \frac{n-1}{2}.$$