Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Junior Section, Round 2)

Saturday, 25 June 2011 **0930-1230**

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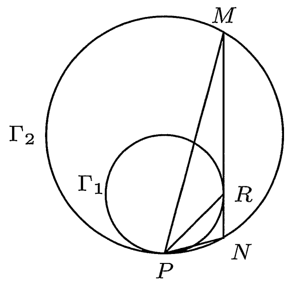
1. Suppose $a, b, c, d > 0$ and $x = \sqrt{a^2 + b^2}$, $y = \sqrt{c^2 + d^2}$. Prove that
$$xy \geq ac + bd.$$

2. Two circles Γ_1, Γ_2 with radii r_1, r_2, respectively, touch internally at the point P. A tangent parallel to the diameter through P touches Γ_1 at R and intersects Γ_2 at M and N. Prove that PR bisects $\angle MPN$.



3. Let $S_1, S_2, \ldots, S_{2011}$ be nonempty sets of consecutive integers such that any 2 of them have a common element. Prove that there is an integer that belongs to every S_i, $i = 1, \ldots, 2011$. (For example, $\{2, 3, 4, 5\}$ is a set of consecutive integers while $\{2, 3, 5\}$ is not.)

4. Any positive integer n can be written in the form $n = 2^a q$, where $a \geq 0$ and q is odd. We call q the *odd part* of n. Define the sequence $a_0, a_1, \ldots,$ as follows: $a_0 = 2^{2011} - 1$ and for $m \geq 0$, a_{m+1} is the odd part of $3a_m + 1$. Find a_{2011}.

5. Initially, the number 10 is written on the board. In each subsequent moves, you can either (i) erase the number 1 and replace it with a 10, or (ii) erase the number 10 and replace it with a 1 and a 25 or (iii) erase a 25 and replace it with two 10. After sometime, you notice that there are exactly one hundred copies of 1 on the board. What is the least possible sum of all the numbers on the board at that moment?