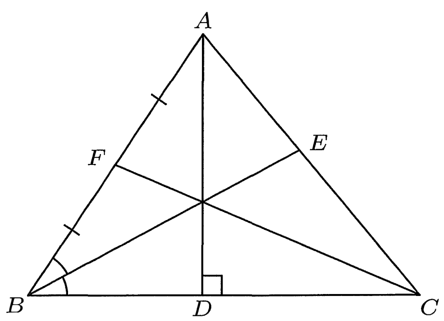
# Singapore Mathematical Society

## Singapore Mathematical Olympiad (SMO) 2011

### (Senior Section, Round 2 solutions)

**1.** There is an error in this problem. The triangle is not necessarily equilateral. In fact we shall prove that the altitude at $A$, the bisector of $\angle B$ and the median at $C$ meet at a common point if and only if $\cos B = \frac{a}{a+c}$ where $BC = a$, $CA = b$ and $AB = c$.

Let $D, E$ and $F$ be the points on $BC, CA$ and $AB$ respectively such that $AD$ is the altitude at $A$, $BE$ is the bisector of $\angle B$ and $CF$ is the median at $C$. Suppose that $AD, BE, CF$ meet at a common point. The point of concurrence of $AD, BE$ and $CF$ must lie inside the triangle $ABC$. Since $F$ is the midpoint of $AB$, by Ceva's theorem $CE : EA = CD : DB$. Using the angle bisector theorem, $CE : EA = a : c$. Thus $CD = a^2/(a+c)$ and $DB = ac/(a+c)$. Thus $\cos B = \frac{BD}{AB} = \frac{a}{a+c}$.




Conversely, if $\cos B = \frac{a}{a+c}$, then $\angle B$ is acute and $BD = c\cos B = ac/(a+c) < a$ so that $D$ is within $BC$. Thus $DC = a - ac/(a+c) = a^2/(a+c)$. Therefore $BD/DC = c/a$. Consequently $(AF/FB)(BD/DC)(CE/EA) = 1$. By Ceva's theorem, $AD, BE$ and $CF$ are concurrent.

So given $a$ and $c$, the acute angle $B$ and hence the triangle $ABC$ is determined. If $a \neq c$, then the triangle $ABC$ is not equilateral.

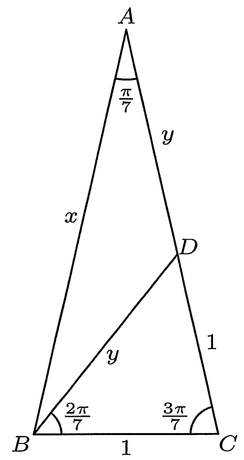
**2.** Yes, in fact, for any $k \in \mathbb{N}$, there is a set $S_k$ having $k$ elements satisfying the given condition. For $k = 1$, let $S_1$ be any singleton set. For $k = 2$, let $S_2 = \{2, 3\}$. Suppose that $S_k = \{a_1, \ldots, a_k\}$ satisfies the given conditions. Let

$$b_1 = a_1 a_2 \cdots a_k$$
$$b_j = b_1 + a_{j-1}, \quad 2 \le j \le k+1.$$

Let $S_{k+1} = \{b_1, b_2, \ldots, b_{k+1}\}$. Then the fact that $S_{k+1}$ satisfies the required property can be verified by observing that $|m - n| = (m, n)$ if and only if $(m - n)$ divides $m$.

**3.** We shall show that $n = 3$ or $7$. Let $f(n) = \cos\frac{\pi}{n}\cos\frac{2\pi}{n}\cos\frac{3\pi}{n}$. One can verify that $f(1) = 1, f(2) = 0, f(3) = \frac{1}{4}, f(4) = 0, f(5) = -\cos^2\frac{2\pi}{5}\cos\frac{\pi}{5} < 0, f(6) = 0$ and $f(8) = \frac{1}{4}$. We shall show that $f(7) = \frac{1}{8}$.

Let $ABC$ be an isosceles triangle with $\angle A = \frac{\pi}{7}$, $\angle B = \angle C = \frac{3\pi}{7}$, $BC = 1$ and $AB = AC = x$. Let $D$ be the point on $AC$ such that $\angle CBD = \frac{2\pi}{7}$. Let $BD = y$. Then the triangles $BCD$ and $ADB$ are isosceles with $BC = CD = 1$ and $AD = BD = y$. Thus $\cos\frac{\pi}{7} = \cos A = \frac{x}{2y}$, $\cos\frac{2\pi}{7} = \cos\angle CBD = \frac{y}{2}$, and $\cos\frac{3\pi}{7} = \cos C = \frac{1}{2x}$. Therefore, $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$.




Lastly, let's show that $f(n) \neq \frac{1}{n+1}$ for $n \geq 9$. For $n \geq 9$, we have $0 < \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n} < \frac{\pi}{2}$. Since cosine is a decreasing function on $[0, \frac{\pi}{2}]$, we have $f(n)$ is an increasing function of $n$ for $n \geq 9$. Consequently, $f(n) \geq f(9) > \cos^3\frac{3\pi}{9} = \frac{1}{8} > \frac{1}{n+1}$.

**4.** Let $a_i = \max S_i$. Without loss of generality, assume that $a_1 \leq a_i$ for all $i$. We shall prove by induction on $k$. For $k = 2$, since $S_1 \cap S_2 \neq \emptyset$, $a_1 \in S_2$. Therefore $X = \{a_1\}$ works. Now assume that the result is true for $k-1$. Let $\mathbb{I}$ be the collection consisting of $S_1$ and the sets $S_i$ such that $S_i \cap S_1 \neq \emptyset$ and let $\mathbb{J}$ be the collection of the other sets. Note that $a_1$ is contained in all the sets in $\mathbb{I}$. If $|\mathbb{J}| < k-1$, then the set $X$ consisting of one integer from each of the sets in $\mathbb{J}$ together with $a_1$ has the desired property. Otherwise, consider a collection $\mathbb{K}$ of $k-1$ sets in $\mathbb{J}$. $\mathbb{K}$, together with $S_1$, forms a collection of $k$ sets. Among these there are two that have nonempty intersection. Since $S_1$ does not intersect any of the sets in $\mathbb{J}$, these two sets must come from $\mathbb{K}$. Thus by the induction hypothesis, there is a set $X'$ of $k-2$ integers such that every set in $\mathbb{J}$ contains one integer in $X'$. Thus $X = X' \cup \{a_1\}$ has the desired property.

**5.** Dividing each of the numerator and denominator of LHS by $2x_1x_2$, $2x_2x_3$, ..., writing $a_1 = \frac{x_3x_4}{x_1x_2}$, $a_2 = \frac{x_4x_5}{x_2x_3}$, ..., and noting that $x_i^2 + x_{i+1}^2 \geq 2x_ix_{i+1}$, we get

$$2 \times \text{LHS} \leq \frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n}.$$

Note that $a_1 a_2 \cdots a_n = 1$. It suffices to show that

$$\frac{a_1}{1+a_1} + \frac{a_2}{1+a_2} + \cdots + \frac{a_n}{1+a_n} \geq 1 \tag{$*$}$$

since it is equivalent to

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n} \leq n-1.$$

We shall show that $(*)$ is true for $n \geq 2$. The case $n = 2$ is obvious. We will now prove it by induction. Suppose $(*)$ holds for $n = k$. Now assume $a_1 \cdots a_{k+1} = 1$, $a_i > 0$ for all $i$. To prove the inductive step, it suffices to show that

$$\frac{a_k}{1+a_k} + \frac{a_{k+1}}{1+a_{k+1}} \geq \frac{a_k a_{k+1}}{1+a_k a_{k+1}}.$$

which can be verified directly.

Note: This is an extension of the problem :

$$\frac{x_1^2}{x_1^2 + x_2 x_3} + \frac{x_2^2}{x_2^2 + x_3 x_4} + \cdots + \frac{x_{n-1}^2}{x_{n-1} + x_n x_1} + \frac{x_n^2}{x_n^2 + x_1 x_2} \leq n-1.$$