Singapore Mathematical Society

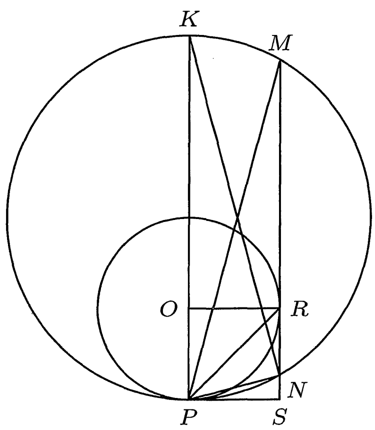
Singapore Mathematical Olympiad (SMO) 2011

(Junior Section, Round 2 solutions)

1. It's equivalent to prove $x^2y^2 \geq (ac+bd)^2$ as all the numbers are nonnegative. This is true since

$$\begin{aligned} x^2y^2 &= (a^2+b^2)(c^2+d^2) \\ &= (ac)^2+(bd)^2+a^2d^2+b^2c^2 \\ &\geq (ac)^2+(bd)^2+2(ac)(bd) \qquad \text{AM-GM} \\ &= (ac+bd)^2. \end{aligned}$$

2. Let the tangent at P meet the tangent at R at the point S. Let O be the centre of Γ_1. Then $ORST$ is a square. Hence $\angle KPR = \angle RPS = 45°$. Also $\angle NPS = \angle NKP = \angle PMS = \angle MPK$. Thus $\angle MPR = \angle RPN$.



3. Let $a_i = \max S_i$, $b_i = \min S_i$ and suppose that $t_1 = \min\{t_i\}$. For each j, if $S_1 \cap S_j \neq \emptyset$, then $a_1 \geq b_j$. Therefore $a_1 \in S_j$.

Note: Problem 4 in the Senior Section is the general version.

4. Replace 2011 by any positive odd integer n. We first show by induction that $a_m = 3^m 2^{n-m} - 1$ for $m = 0, 1, \ldots, n-1$. This is certainly true for $m = 0$. Suppose it's true for some $m < n-1$. Then $3a_m + 1 = 3^{m+1}2^{n-m} - 2$. Since $n - m > 1$, the odd part is $3^{m+1}2^{n-m-1} - 1$ which is a_{m+1}. Now $a_{n-1} = 3^{n-1}2^1 - 1$. Thus

$3a_{n-1}+1=3^n2-2=2(3^n-1)$. When n is odd, $3^n \equiv -1 \pmod 4$. Thus $4 \nmid 3^n-1$. Hence the odd part of $2(3^n-1)$ is $\frac{3^n-1}{2}$ and this is the value of a_n.

5. Suppose the number of times that operations (i), (ii) and (iii) have been performed are x, y and z, respectively. Then the number of 1, 10 and 25 are $y-x$, $1+x-y+2z$ and $y-z$, respectively, with $-x+y=100$. Thus the sum is

$$S = y-x+10(1+x-y+2z)+25(y-z) = -890+5(5y-z).$$

Since we want the minimum values of S, y has to be as small as possible and z as large as possible. Since

$$y-x=100,\ 1+x-y+2z \geq 0, \quad y-z \geq 0$$

we get, from the first equation, $y \geq 100$, from the second inequality, $2z \geq 99$ or $z \geq 50$ and $y \geq z$ from the third. Thus the minimum is achieved when $y=100$, $x=0$ and $z=100$. Thus minimum $S=1100$.