Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Senior Section, Round 2)

Saturday, 29 June 2013

1. In the triangle ABC, $AB > AC$, the extension of the altitude AD with D lying inside BC intersects the circumcircle ω of the triangle ABC at P. The circle through P and tangent to BC at D intersects ω at Q distinct from P with $PQ = DQ$. Prove that $AD = BD - DC$.

2. Find all pairs of integers (m, n) such that

$$m^3 - n^3 = 2mn + 8.$$

3. Let $b_1, b_2, \ldots$ be a sequence of positive real numbers such that for each $n \geq 1$,

$$b_{n+1}^2 \geq \frac{b_1^2}{1^3} + \frac{b_2^2}{2^3} + \cdots + \frac{b_n^2}{n^3}.$$

   Show that there is a positive integer M such that

$$\sum_{n=1}^{M} \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} > \frac{2013}{1013}.$$

4. In the following 6×6 array, one can choose any $k \times k$ subarray, with $1 < k \leq 6$ and add 1 to all its entries. Is it possible to perform the operation a finite number of times so that all the entries in the array are multiples of 3?

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5. Let x, y be distinct real numbers such that $\frac{x^n - y^n}{x - y}$ is an integer for four consecutive positive integers n. Prove that $\frac{x^n - y^n}{x - y}$ is an integer for all positive integers n.