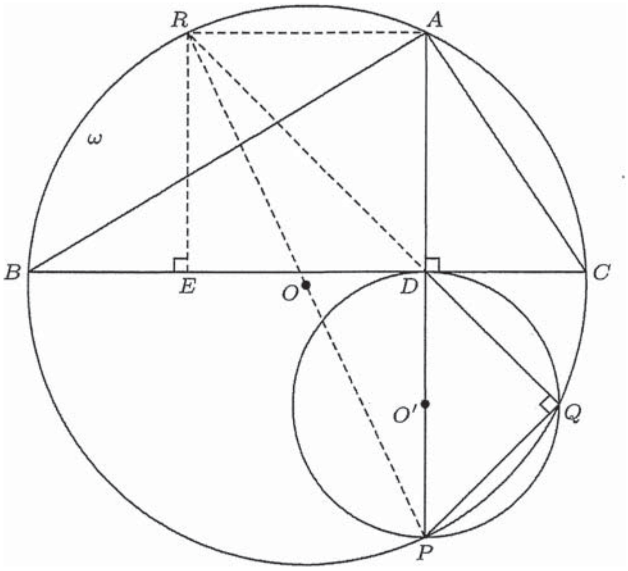
# Singapore Mathematical Society

## Singapore Mathematical Olympiad (SMO) 2013

### (Senior Section, Round 2 solutions)

**1.**




Let the extension of $QD$ meet $\omega$ at $R$. Since $\angle PQR = 90°$, $PR$ is a diameter of $\omega$. Thus $\angle PAR = 90°$ so that $RA$ is parallel to $BC$. This means $BCAR$ is an isosceles trapezoid. Let $E$ be the foot of the perpendicular from $R$ onto $BC$. Then $BE = CD$ and $ARED$ is a rectangle. Since $\angle ADR = 45°$, $ARED$ is in fact a square so that $AD = DE$. Therefore, $BD - DC = BD - BE = DE = AD$.

**2.** When $m = 0$, $n = -2$ and when $n = 0$, $m = 2$. These are the two obvious solutions: $(m, n) = (0, -2), (2, 0)$. We'll show that there are no solutions when $mn \neq 0$.

Suppose $mn < 0$. If $m > 0, n < 0$, then

$$-2m|n| + 8 = m^3 + |n|^3 \geq m^2 + |n|^2 \quad \Rightarrow \quad 8 \geq (m + |n|)^2.$$

Thus $m + |n| = 2$ or $m = 1$, $n = -1$ and this is not a solution. If $m < 0$ and $n > 0$, then

$$-2|m|n + 8 = -|m|^3 - n^3 \leq -|m|^2 - |n|^2 \quad \Rightarrow \quad 8 \leq -(|m| - n)^2$$

which is impossible.

**3.** From Cauchy -Schwarz inequality, we have

$$\left(\sum_{k=1}^{n}\frac{b_k^2}{k^3}\right)\left(\sum_{k=1}^{n}k^3\right)\geq\left(\sum_{k=1}^{n}b_k^2\right)^2$$
$$\Rightarrow\quad \left(\sum_{k=1}^{n}\frac{b_k^2}{k^3}\right)\left(\frac{n(n+1)}{2}\right)^2\geq\left(\sum_{k=1}^{n}b_k^2\right)^2$$
$$\Rightarrow\quad \frac{b_{n+1}}{b_1+b_2+\cdots+b_n}\geq\frac{2}{n(n+1)}$$
$$\Rightarrow\quad \sum_{n=1}^{M}\frac{b_{n+1}}{b_1+b_2+\cdots+b_n}\geq\sum_{n=1}^{M}\frac{2}{n(n+1)}$$
$$\Rightarrow\quad \sum_{n=1}^{M}\frac{b_{n+1}}{b_1+b_2+\cdots+b_n}\geq 2\sum_{n=1}^{M}\frac{1}{n}-\frac{1}{n+1}$$
$$=2-\frac{2}{M+1}\geq\frac{2013}{1013}$$

if $M \geq 155$.

**4.** The answer is no. Let the original array be $A$. Consider the following array

$$M=\begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0\\ -1 & 0 & 1 & -1 & 0 & 1\\ -1 & -1 & 0 & 0 & 1 & 1\\ 1 & 1 & 0 & 0 & -1 & -1\\ 1 & 0 & -1 & 1 & 0 & -1\\ 0 & -1 & -1 & 1 & 1 & 0\end{bmatrix}.$$

Multiply the corresponding elements of the two arrays and compute the sum modulo 3. It's easy to verify that this sum is invariant under the given operation. Since the original sum is 2, one can never obtain an array where all the entries are multiples of 3.

**5.** Note that

$$(x+y)(x^n-y^n)=(x^{n+1}-y^{n+1})+xy(x^{n-1}-y^{n-1}).$$

Let $t_n=\frac{x^n-y^n}{x-y}$. Then $t_0=0$, $t_1=1$ and for $n\geq 0$, $t_{n+2}+bt_{n+1}+ct_n=0$, with $b=-(x+y)$ and $c=xy$. Then it suffices to show that $b,c\in\mathbb{Z}$.

If $c=0$, then either $x=0$ or $y=0$. Say $y=0$. Then $t_n=x^{n-1}$, with $x\neq 0$. Then $x=\frac{t_{m+1}}{t_m}\in\mathbb{Q}$. From $t_{m+1}=x^m\in\mathbb{Z}$, it follows that $x\in\mathbb{Z}$. Thus $t_n\in\mathbb{Z}$ for all $n$. The case $x=0$ is similar.

We now assume that $c \neq 0$. Let $t_n \in \mathbb{Z}$ for $n = m, m+1, m+2, m+3$. Note that $c^n = (xy)^n = t_{n+1}^2 - t_n t_{n+2}$. Thus $c^m, c^{m+1} \in \mathbb{Z}$. Therefore $c = \frac{c^{m+1}}{c^m} \in \mathbb{Q}$. As before, we have $c \in \mathbb{Z}$. If both $t_{m+1}, t_{m+2}$ are 0, then using the recurrence, we can show easily that $t_n = 0$ for all $n$, a contradiction. Thus one of them is nonzero. Note that, with $k = m+1$ or $m+2$, whichever is nonzero, we have

$$b = \frac{-ct_{k-1} - t_{k+1}}{t_k} \in \mathbb{Q}.$$

From the recurrence, it follows by induction that $t_n$ can be represented as $t_n = f_{n-1}(b)$ where $f_{n-1}(X)$ is a polynomial with integer coefficients, $\deg f_{n-1} = n-1$ and with the coefficient of $X^{n-1} = \pm 1$. Since $b \in \mathbb{Q}$ is a root of the equation $f_m(X) = t_{m+1}$, $b \in \mathbb{Z}$.