Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Junior Section, Round 2 solutions)

1. We have 37 is either $a+d$ or $b+c$ and

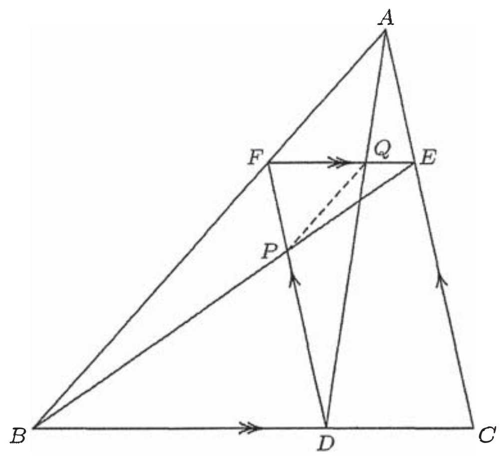
$$a+b=32, \quad a+c=36, \quad c+e=48, \quad d+e=51$$

Thus $c-b=4$, $d-c=3$ and $d-b=7$. Therefore $(a+b)+(d-b)=a+d=39$. Hence $b+c=37$. We thus have $a=15.5$, $b=16.5$, $c=20.5$, $d=23.5$ and $e=27.5$.

2. Since FE is parallel to BC and DF is parallel to CA, we have the triangles PFE, PDB and ECB are similar. Also the triangles AFQ and ABD are similar, FBD and ABC are similar. It follows that

$$\frac{DP}{PF}=\frac{BP}{PE}=\frac{BD}{DC}=\frac{BF}{FA}=\frac{DQ}{QA}$$

so that PQ is parallel to AB.



3. Let p be such a prime, then $p>2$ and is therefore odd. Thus $p=q-2=r+2$ where q,r are primes. If $r\equiv 1 \pmod 3$, then $p\equiv 0 \pmod 3$ and therefore $p=3$ and $r=1$ which is impossible. If $r\equiv 2 \pmod 3$, then $q\equiv 0 \pmod 3$ and thus $q=3$ and so $p=1$, again impossible. Thus $r\equiv 0 \pmod 3$, which means $r=3$ and hence $p=5$ and $q=7$. Thus $p=5$ is the only such prime.

4. We have $a = bm + r$ where $m = \lfloor a/b \rfloor$ and $0 \le r < b$. Thus

$$\frac{2^a+1}{2^b-1} = \frac{2^a-2^r}{2^b-1} + \frac{2^r+1}{2^b-1}.$$

Note that $2^a - 2^r = 2^r(2^{a-r} - 1) = 2^r(2^{bm} - 1)$, and

$$2^{bm} - 1 = (2^b)^m - 1 = (2^b - 1)[(2^b)^{m-1} + (2^b)^{m-2} + \cdots + 1].$$

Therefore $\frac{2^a-2^r}{2^b-1}$ is an integer.

Observe that if $b > 2$, then $2^{b-1}(2-1) > 2$, i.e.,

$$2^r + 1 \le 2^{b-1} + 1 < 2^b - 1.$$

Therefore $\frac{2^r+1}{2^b-1}$ is not an integer. Thus $\frac{2^a+1}{2^b-1}$ is not an integer.

5. Let the musicians be A, B, C, D, E, F. We first show that four concerts are sufficient. The four concerts with the performing musicians: $\{A, B, C\}$, $\{A, D, E\}$, $\{B, D, F\}$ and $\{C, E, F\}$ satisfy the requirement. We shall now prove that 3 concerts are not sufficient. Suppose there are only three concerts. Since everyone must perform at least once, there is a concert where two of the musicians, say A, B, played. But they must also play for each other. Thus we have A played and B listened in the second concert and vice versa in the third. Now C, D, E, F must all perform in the second and third concerts since these are the only times when A and B are in the audience. It is not possible for them to perform for each other in the first concert. Thus the minimum is 4.