Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2020

Junior Section (Round 1)

Monday, 7 September 2020

0930-1200 hrs
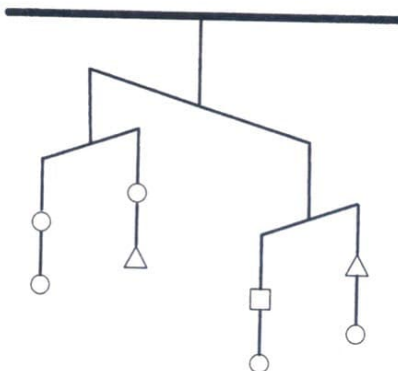
Instructions to contestants

1. *Answer ALL 25 questions.*

2. *Enter your answers on the answer sheet provided.*

3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*

4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*

5. *No steps are needed to justify your answers.*

6. *Each question carries 1 mark.*

7. *No calculators are allowed.*

8. *Throughout this paper, let $[A_1 A_2 \ldots A_r]$ denote the area enclosed by the polygon $A_1 A_2 \ldots A_r$.*

9. *Throughout this paper, let $\overline{a_{n-1}a_{n-2}\ldots a_0}$ denote an n-digit number with the digits a_i in the corresponding position, i.e. $\overline{a_{n-1}a_{n-2}\ldots a_0} = a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + \cdots + a_0 10^0.$*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Let $x = 2^{300}$, $y = 3^{200}$ and $z = 6^{100}$. Which of the following is true?

   (A) $x > y > z$ (B) $x > z > y$ (C) $y > z > x$ (D) $y > x > z$ (E) $z > x > y$

2. An expensive painting was stolen and the police rounded up five suspects Alfred, Boris, Chucky, Dan and Eddie. These were the statements that were recorded.
   Alfred: "Either Boris or Dan stole the painting."
   Boris: "I think Dan or Eddie is the guilty party."
   Chucky: "It must be Dan."
   Dan: "Boris or Eddie did it!"
   Eddie: "I am absolutely sure the thief is Alfred."
   The police knew that only one of the five suspects stole the painting and that all five were lying. Who stole the painting?

   (A) Alfred   (B) Boris   (C) Chucky   (D) Dan   (E) Eddie

3. The following diagram shows a system of balances hanging from the ceiling with three types of weights. The balances tip down to the heavier side. If we use $\square < \triangle$ to represent $\square$ is lighter than $\triangle$, which of the following is true?

   (A) $\square < \bigcirc < \triangle$ (B) $\square < \triangle < \bigcirc$ (C) $\triangle < \square < \bigcirc$ (D) $\triangle < \bigcirc < \square$
   (E) $\bigcirc < \square < \triangle$

4. The integer 6 has exactly four positive factors, namely $1, 2, 3$ and 6. Likewise, the integer 8 has exactly four positive factors, $1, 2, 4$ and 8. How many integers from 9 to 50 (inclusive) have exactly four positive factors?

   (A) 10   (B) 11   (C) 12   (D) 13   (E) 14

5. A company sells a bottle of juice at $1 per bottle and offers to give you a free bottle of juice if you recycle four empty bottles. Assuming there is no limit to the offer and you can recycle bottles that you receive from the offer, what is the largest number of bottles you can eventually consume if you have $43?

   (A) 53   (B) 54   (C) 55   (D) 56   (E) 57

2

Short Questions

6. Let n be a positive integer. Suppose the lowest common multiple of 4, 5 and n is 2020. What is the sum of the smallest possible value of n and the largest possible value of n?

7. When the five-digit integer $\overline{2x6yx}$ is divided by the four-digit integer $\overline{5y27}$, the quotient is 4 and remainder is $\overline{x106}$, which is a four-digit integer. What is the value of the digit x?

8. How many multiples of 11 are there in the sequence $1100, 1101, 1102, \ldots, 2019, 2020$?

9. A quadruple (a, b, c, d) of positive integers is *skewed* if the median and mode of a, b, c, d are equal, but strictly greater than the mean of a, b, c, d. How many skewed (a, b, c, d) of positive integers are there that satisfy $a \le b \le c \le d$ and $a + b + c + d = 40$?

10. Let T_n be a linear sequence given by

$$T_1 = 10, \ T_2 = 13, \ T_3 = 16, \ldots,$$

and S_n be a quadratic sequence given by

$$S_1 = 10, \ S_2 = 23, \ S_3 = 39, \ldots.$$

What is the value of $S_{101} - T_{101}$?

11. Let ABC be a triangle where D is the midpoint of BC and E lies on AC such that $AE : EC = 3 : 1$. Let F be the intersection of AD and BE. If the area of ABC is 280, what is the area of triangle BFD?

12. In the figure below, each distinct letter represents a unique distinct digit such that the arithmetic holds. If the letter K represents 6, what number does SHAKE represent?

$$
\begin{array}{ccccccc}
 & S & H & A & K & E \\
+ & S & H & A & K & E \\
\hline
R & A & T & T & L & E \\
\end{array}
$$

13. The diagram below shows a shape formed from four semi-circles whose radii are distinct integers. If the shape has perimeter of 18π and encloses an area of $k\pi$, where k is a prime, what is the value of k?
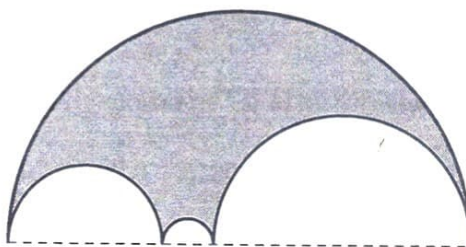
14. If the sum of the two diagonals of a kite is 100, what is the largest possible area of the kite?

15. What is the smallest five-digit integer of the form $\overline{5x20y}$ that is divisible by 33?

16. What is the value of $2020^3 - 9(2020)(2017) - 2017^3$?

17. In the following diagram, $ABCD$ is a rectangle where E and F are points on BC and CD respectively. The area of triangle AEF, denoted $[AEF]$, is 2037. If

$$[AECF] = 2[ABE] = 3[ADF],$$

what is the area of the rectangle $ABCD$?
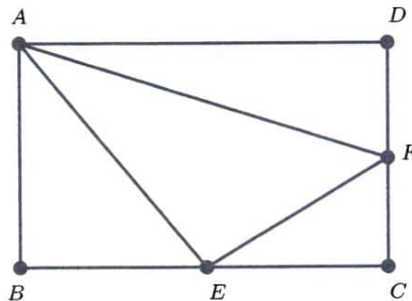
18. In the following diagram, $ABCD$ is a square of side length 64. E is the midpoint of AB, F is the midpoint of EC and G is the midpoint of FD. What is the area enclosed by the quadrilateral $AEFG$?
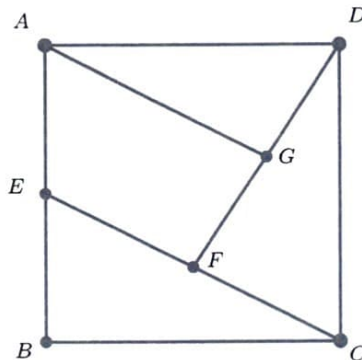
19. Let $X = 1234\cdots78798081$ be the integer that consists of all the integers from 1 to 81 written from left to right. What is the remainder of X when divided by 2020?

20. Let $A = \frac{1}{7} \times 3.14\dot{1}\dot{5}$, where $3.14\dot{1}\dot{5}$ is the rational number with recurring digits 15. In other words,

$$3.14\dot{1}\dot{5} = 3.14 + 0.0015 + 0.000015 + 0.00000015 + \cdots.$$

Suppose that $A = \frac{m}{n}$, where m and n are positive integers with no common factors larger than 1. What is the value of $m + n$?

21. Ali and Barry went running on a standard 400 metre track. They started simultaneously at the same location on the track but ran in opposite directions. Coincidentally, after 24 minutes, they ended at the same location where they started. Ali completed 12 rounds of the track in those 24 minutes while Barry completed 10 rounds. How many times did Ali and Barry pass each other during the run? (Exclude from your answer the times that they met at the start of the of run and when they completed the run after 24 minutes.)

22. If we have

$$(1 - 3x) + (1 - 3x)^2 + \cdots + (1 - 3x)^{100} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{100} x^{100},$$

for some integers $a_0, a_1, \ldots, a_{100}$, what is the value of

$$\left| \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_{100}}{3^{100}} \right|?$$

23. What is the value of $\sqrt{27 - \sqrt{704}} + \sqrt{27 + \sqrt{704}}$?

24. If $m > n$ are positive integers satisfying

$$(m^2 - n^2)^2 = 1 + 80n,$$

what is the smallest possible value of mn?

25. What is the value of

$$\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{37} \right) + \left(\frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \cdots + \frac{2}{37} \right)$$
$$+ \left(\frac{3}{4} + \frac{3}{5} + \frac{3}{6} + \cdots + \frac{3}{37} \right) + \cdots + \left(\frac{35}{36} + \frac{35}{37} \right) + \frac{36}{37} ?$$