SMO Open 2024 Rd.2

Question 1

In triangle ABC, $\angle B = 90^\circ$, $AB > BC$, and P is the point such that $BP = PC$ and $\angle APB = 90^\circ$ where P and C lie on the same side of AB. Let Q be the point on AB such that $AP = AQ$, and let M be the midpoint of QC. Prove that the line through M parallel to AP passes through the midpoint of AB.

Question 2

Find the minimum value of

$$\frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n}$$

where $x_1, x_2, \ldots, x_n$ are distinct positive integers.

Question 3

Prove that for every positive integer n, there is a unique n-digit integer $A(n)$ which is a multiple of 5^n and whose digits are all odd.

Question 4

Alice and Bob play a game. Bob starts by picking a set S consisting of M vectors of length n with entries either 0 or 1. Alice picks a sequence of numbers $y_1 \leq y_2 \leq \cdots \leq y_n$ from the interval $[0, 1]$, and a choice of real numbers $x_1, \ldots, x_n \in \mathbb{R}$. Bob wins if he can pick a vector $(z_1, \ldots, z_n) \in S$ such that

$$\sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} x_i z_i,$$

otherwise Alice wins. Determine the minimum value of M so that Bob can guarantee a win.

Question 5

Let p be a prime number. Determine the largest possible n such that the following holds. It is possible to fill an $n \times n$ table with integers a_{ik} in the i-th row and k-th column, for $1 \leq i, k \leq n$, such that for any quadruple i, j, k, l with $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, the number $a_{ik}a_{jl} - a_{il}a_{jk}$ is not divisible by p.