Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Senior Section)

Tuesday, 31 May 2011 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Suppose a, b and c are nonzero real numbers such that
$$\frac{a^2}{b^2+c^2} < \frac{b^2}{c^2+a^2} < \frac{c^2}{a^2+b^2}.$$
Which of the following statements is always true?

(A) $a < b < c$ (B) $|a| < |b| < |c|$ (C) $c < b < a$
(D) $|b| < |c| < |a|$ (E) $|c| < |b| < |a|$

2. Suppose θ is an angle between 0 and $\frac{\pi}{2}$, and $\sin 2\theta = a$. Which of the following expressions is equal to $\sin\theta + \cos\theta$?

(A) $\sqrt{a+1}$ (B) $(\sqrt{2}-1)a+1$ (C) $\sqrt{a+1}-\sqrt{a^2-a}$
(D) $\sqrt{a+1}+\sqrt{a^2-a}$ (E) None of the above

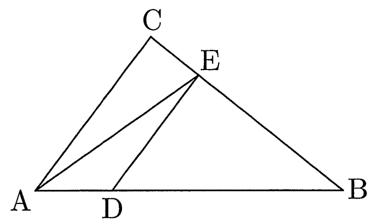
3. Let x be a real number. If $a = 2011x + 9997$, $b = 2011x + 9998$ and $c = 2011x + 9999$, find the value of $a^2 + b^2 + c^2 - ab - bc - ca$.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

4. Suppose x, y are real numbers such that $\frac{1}{x} - \frac{1}{2y} = \frac{1}{2x+y}$. Find the value of $\frac{y^2}{x^2} + \frac{x^2}{y^2}$.

(A) $\frac{2}{3}$ (B) $\frac{9}{2}$ (C) $\frac{9}{4}$ (D) $\frac{4}{9}$ (E) $\frac{2}{9}$

5. In the figure below, ABC is a triangle, and D and E are points on AB and BC respectively. It is given that DE is parallel to AC, and $CE : EB = 1 : 3$. If the area of $\triangle ABC$ is 1440 cm^2 and the area of $\triangle ADE$ is x cm^2, what is the value of x?



(A) 288 (B) 240 (C) 320 (D) 384 (E) 270

6. Determine the value of

$$\frac{2}{\dfrac{1}{\sqrt{2}+\sqrt[4]{8}+2}+\dfrac{1}{\sqrt{2}-\sqrt[4]{8}+2}}.$$

(A) $4-\sqrt{2}$ (B) $2-2\sqrt{2}$ (C) $4+\sqrt{2}$ (D) $2\sqrt{2}+4$
(E) $4\sqrt{2}-2$

7. Let $x=\dfrac{1}{\log_{\frac{1}{3}}\frac{1}{2}}+\dfrac{1}{\log_{\frac{1}{5}}\frac{1}{4}}+\dfrac{1}{\log_{\frac{1}{7}}\frac{1}{8}}$. Which of the following statements is true?

(A) $1.5<x<2$ (B) $2<x<2.5$ (C) $2.5<x<3$
(D) $3<x<3.5$ (E) $3.5<x<4$

8. Determine the last two digits of 7^{5^6}.

(A) 01 (B) 07 (C) 09 (D) 43 (E) 49

9. It is given that x and y are two real numbers such that $x>1$ and $y>1$. Find the smallest possible value of

$$\frac{\log_x 2011+\log_y 2011}{\log_{xy} 2011}.$$

(A) 4 (B) 6 (C) 8 (D) 10 (E) 12

10. It is given that a, b and c are three real numbers such that $a+b=c-1$ and $ab=c^2-7c+14$. Find the largest possible value of a^2+b^2.

(A) 5 (B) 6 (C) 8 (D) 9 (E) 10

Short Questions

11. Find the value of

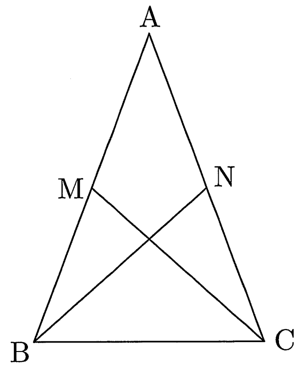
$$\frac{2011^2+2111^2-2\times 2011\times 2111}{25}.$$

12. Find the largest natural number n which satisfies the inequality

$$n^{6033} < 2011^{2011}.$$

13. Find the integer which is closest to $\dfrac{(1+\sqrt{3})^4}{4}$.

14. In the diagram below, $\triangle ABC$ is an isosceles triangle with $AB = AC$, and M and N are the midpoints of AB and AC respectively. It is given that CM is perpendicular to BN, $BC = 20$ cm, and the area of $\triangle ABC$ is x cm^2. Find the value of x.



15. Find the smallest positive integer n such that

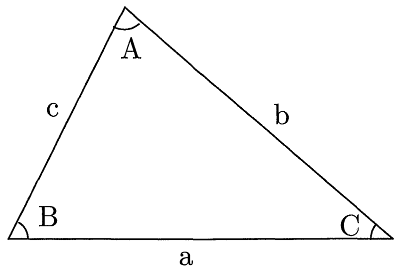
$$\sqrt{5n} - \sqrt{5n-4} < 0.01.$$

16. Find the value of

$$\frac{1}{1+11^{-2011}} + \frac{1}{1+11^{-2009}} + \frac{1}{1+11^{-2007}} + \cdots + \frac{1}{1+11^{2009}} + \frac{1}{1+11^{2011}}.$$

17. Let $x = \sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{7\pi}{8}\right) + \cos^4\left(\frac{7\pi}{8}\right)$. Find the value of $36x$.

18. In the diagram below, the lengths of the three sides of the triangle are a cm, b cm and c cm. It is given that $\dfrac{a^2+b^2}{c^2}=2011$. Find the value of $\dfrac{\cot C}{\cot A+\cot B}$.



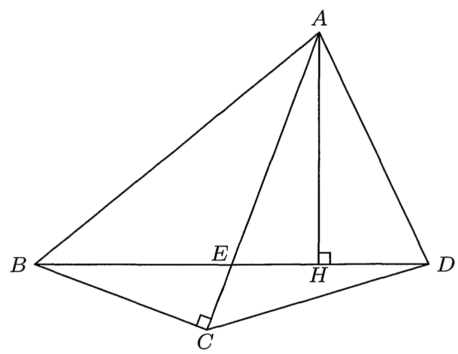
19. Suppose there are a total of 2011 participants in a mathematics competition, and at least 1000 of them are female. Moreover, given any 1011 participants, at least 11 of them are male. How many male participants are there in this competition?

20. Let $f:\mathbb{Q}\setminus\{0,1\}\to\mathbb{Q}$ be a function such that

$$x^2f(x)+f\left(\frac{x-1}{x}\right)=2x^2$$

for all rational numbers $x\neq 0,1$. Here $\mathbb{Q}$ denotes the set of rational numbers. Find the value of $f(\frac{1}{2})$.

21. In the diagram below, $ABCD$ is a convex quadrilateral such that AC intersects BD at the midpoint E of BD. The point H is the foot of the perpendicular from A onto DE, and H lies in the interior of the segment DE. Suppose $\angle BCA=90^\circ$, $CE=12$ cm, $EH=15$ cm, $AH=40$ cm and $CD=x$ cm. Find the value of x.



22. How many pairs of integers (x, y) are there such that

$$x \geq y \quad \text{and} \quad \frac{1}{x}+\frac{1}{y}=\frac{1}{211}\,?$$

23. It is given that a, b, c are three real numbers such that the roots of the equation $x^2+3x-1=0$ also satisfy the equation $x^4+ax^2+bx+c=0$. Find the value of $a+b+4c+100$.

24. It is given that m and n are two positive integers such that

$$n-\frac{m}{n}=\frac{2011}{3}.$$

Determine the smallest possible value of m.

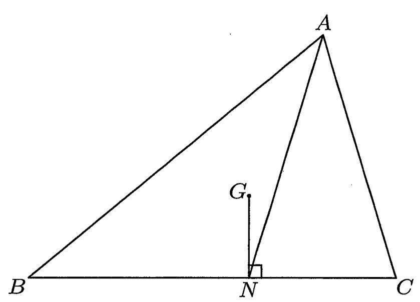
25. It is given that a, b, c are positive integers such that the roots of the three quadratic equations

$$x^2-2ax+b=0, \quad x^2-2bx+c=0, \quad x^2-2cx+a=0$$

are all positive integers. Determine the maximum value of the product abc.

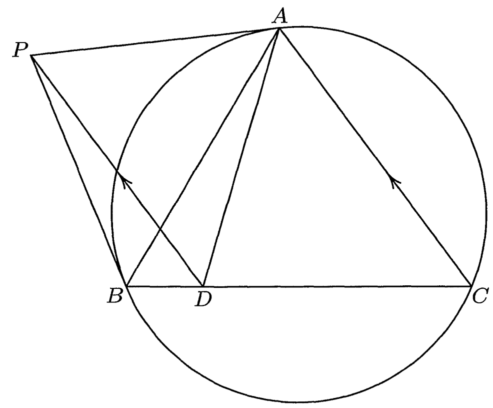
26. Suppose A, B, C are three angles such that $A \geq B \geq C \geq \frac{\pi}{8}$ and $A+B+C=\frac{\pi}{2}$. Find the largest possible value of the product $720 \times (\sin A) \times (\cos B) \times (\sin C)$.

27. In the diagram below, ABC is a triangle such that AB is longer than AC. The point N lies on BC such that AN bisects $\angle BAC$. The point G is the centroid of $\triangle ABC$, and it is given that GN is perpendicular to BC. Suppose $AC=6$ cm, $BC=5\sqrt{3}$ cm and $AB=x$ cm. Find the value of x.



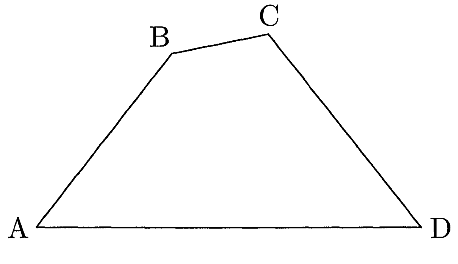
28. It is given that a, b, c and d are four positive prime numbers such that the product of these four prime numbers is equal to the sum of 55 consecutive positive integers. Find the smallest possible value of $a+b+c+d$. (Remark: The four numbers a, b, c, d are not necessarily distinct.)

29. In the diagram below, ABC is a triangle with $AB = 39$ cm , $BC = 45$ cm and $CA = 42$ cm. The tangents at A and B to the circumcircle of $\triangle ABC$ meet at the point P. The point D lies on BC such that PD is parallel to AC. It is given that the area of $\triangle ABD$ is x cm^2. Find the value of x.



30. It is given that a and b are positive integers such that a has exactly 9 positive divisors and b has exactly 10 positive divisors. If the least common multiple (LCM) of a and b is 4400, find the value of $a+b$.

31. In the diagram below, $ABCD$ is a quadrilateral such that $\angle ABC = 135°$ and $\angle BCD = 120°$. Moreover, it is given that $AB = 2\sqrt{3}$ cm, $BC = 4 - 2\sqrt{2}$ cm, $CD = 4\sqrt{2}$ cm and $AD = x$ cm. Find the value of $x^2 - 4x$.



32. It is given that p is a prime number such that

$$x^3 + y^3 - 3xy = p - 1$$

for some positive integers x and y. Determine the largest possible value of p.

33. It is given that a, b and c are three positive integers such that

$$a^2 + b^2 + c^2 = 2011.$$

Let the highest common factor (HCF) and the least common multiple (LCM) of the three numbers a, b, c be denoted by x and y respectively. Suppose that $x + y = 388$. Find the value of $a + b + c$. (Remark: The highest common factor is also known as the greatest common divisor.)

34. Consider the set $S = \{1, 2, 3, \cdots, 2010, 2011\}$. A subset T of S is said to be a k-element RP-subset if T has exactly k elements and every pair of elements of T are relatively prime. Find the smallest positive integer k such that every k-element RP-subset of S contains at least one prime number. (As an example, $\{1, 8, 9\}$ is a 3-element RP-subset of S which does not contain any prime number.)

35. In the diagram below, P is a point on the semi-circle with diameter AB. The point L is the foot of the perpendicular from P onto AB, and K is the midpoint of PB. The tangents to the semicircle at A and at P meet at the point Q. It is given that PL intersects QB at the point M, and KL intersects QB at the point N. Suppose $\dfrac{AQ}{AB} = \dfrac{5}{12}$, $QM = 25$ cm and $MN = x$ cm. Find the value of x.