Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Junior Section)

Tuesday, 31 May 2011 0930–1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer on the answer sheet and shade the appropriate bubbles below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Calculate the following sum:

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots+\frac{10}{2^{10}}.$$

(A) $\frac{503}{256}$; (B) $\frac{505}{256}$; (C) $\frac{507}{256}$; (D) $\frac{509}{256}$; (E) None of the above.

2. It is known that the roots of the equation

$$x^5+3x^4-4044118x^3-12132362x^2-12132363x-2011^2=0$$

are all integers. How many distinct roots does the equation have?

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

3. A fair dice is thrown three times. The results of the first, second and third throw are recorded as x, y and z, respectively. Suppose $x+y=z$. What is the probability that at least one of x, y and z is 2?

(A) $\frac{1}{12}$; (B) $\frac{3}{8}$; (C) $\frac{8}{15}$; (D) $\frac{1}{3}$; (E) $\frac{7}{13}$.

4. Let

$$x = 1\underbrace{000\cdots000}_{2011\text{ times}}1\underbrace{000\cdots000}_{2012\text{ times}}50.$$

Which of the following is a perfect square?

(A) $x-75$; (B) $x-25$; (C) x; (D) $x+25$; (E) $x+75$.

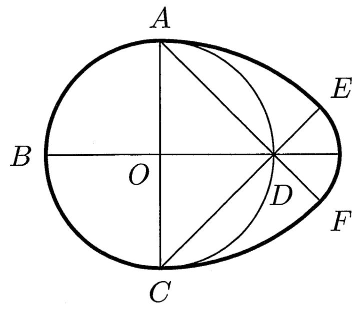
5. Suppose $N_1, N_2, \ldots, N_{2011}$ are positive integers. Let

$$X = (N_1+N_2+\cdots+N_{2010})(N_2+N_3+\cdots+N_{2011}),$$
$$Y = (N_1+N_2+\cdots+N_{2011})(N_2+N_3+\cdots+N_{2010}).$$

Which one of the following relationships always holds?

(A) $X=Y$; (B) $X>Y$; (C) $X<Y$; (D) $X-N_1<Y-N_{2011}$; (E) None of the above.

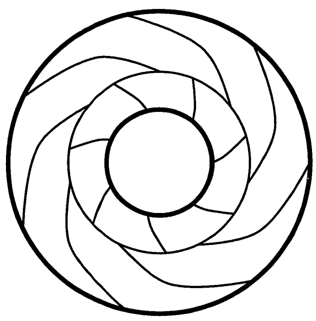
6. Consider the following egg shaped curve. $ABCD$ is a circle of radius 1 centred at O. The arc $\overset{\frown}{AE}$ is centred at C, $\overset{\frown}{CF}$ is centred at A and $\overset{\frown}{EF}$ is centred at D.



What is the area of the region enclosed by the egg shaped curve?

(A) $(3-\sqrt{2})\pi-1$; (B) $(3-\sqrt{2})\pi$; (C) $(3+\sqrt{2})\pi+1$; (D) $(3-2\sqrt{2})\pi$; (E) $(3-2\sqrt{2})\pi+1$.

7. The following annulus is cut into 14 regions. Each region is painted with one colour. What is the minimum number of colours needed to paint the annulus so that any no two adjacent regions share the same colours?



(A) 3; (B) 4; (C) 5; (D) 6; (E) 7.

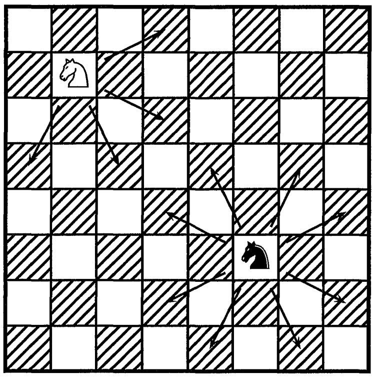
8. Let $n = (2^4-1)(3^6-1)(5^{10}-1)(7^{12}-1)$. Which of the following statements is true?

(A) n is divisible by $5, 7$ and 11 but not 13; (B) n is divisible by $5, 7$ and 13 but not 11;

(C) n is divisible by $5, 11$ and 13 but not 7; (D) n is divisible by $7, 11$ and 13 but not 5;

(E) None of the above.

9. How many ways can you place a White Knight and a Black Knight on an 8×8 chessboard such that they do not attack each other?



(A) 1680; (B) 1712; (C) 3696; (D) 3760; (E) None of the above.

10. In the set $\{1, 6, 7, 9\}$, which of the numbers appear as the last digit of n^n for infinitely many positive integers n?

(A) $1, 6, 7$ only; (B) $1, 6, 9$ only; (C) $1, 7, 9$ only; (D) $6, 7, 9$ only; (E) $1, 6, 7, 9$.

Short Questions

11. Suppose $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \sqrt{2}$ and $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$. Find

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}.$$

12. Suppose $x = \frac{13}{\sqrt{19 + 8\sqrt{3}}}$. Find the exact value of

$$\frac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15}.$$

13. Let $a_1 = 3$, and define $a_{n+1} = \frac{\sqrt{3}\, a_n - 1}{a_n + \sqrt{3}}$ for all positive integers n. Find a_{2011}.

14. Let a, b, c be positive real numbers such that

$$\begin{cases} a^2 + ab + b^2 = 25, \\ b^2 + bc + c^2 = 49, \\ c^2 + ca + a^2 = 64. \end{cases}$$

Find $(a + b + c)^2$.

15. Let $P(x)$ be a polynomial of degree 2010. Suppose $P(n) = \dfrac{n}{1+n}$ for all $n = 0, 1, 2, \ldots, 2010$. Find $P(2012)$.

16. Let $\lfloor x \rfloor$ be the greatest integer smaller than or equal to x. How many solutions are there to the equation $x^3 - \lfloor x^3 \rfloor = (x - \lfloor x \rfloor)^3$ on the interval $[1, 20]$?

17. Let n be the smallest positive integer such that the sum of its digits is 2011. How many digits does n have?

18. Find the largest positive integer n such that $n + 10$ is a divisor of $n^3 + 2011$.

19. Let a, b, c, d be real numbers such that

$$\begin{cases} a^2 + b^2 + 2a - 4b + 4 = 0, \\ c^2 + d^2 - 4c + 4d + 4 = 0. \end{cases}$$

Let m and M be the minimum and the maximum values of $(a-c)^2 + (b-d)^2$, respectively. What is $m \times M$?

20. Suppose $x_1, x_2, \ldots, x_{2011}$ are positive integers satisfying

$$x_1 + x_2 + \cdots + x_{2011} = x_1 x_2 \cdots x_{2011}.$$

Find the maximum value of $x_1 + x_2 + \cdots + x_{2011}$.

21. Suppose that a function $M(n)$, where n is a positive integer, is defined by

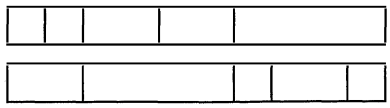
$$M(n) = \begin{cases} n - 10 & \text{if } n > 100, \\ M(M(n+11)) & \text{if } n \leq 100. \end{cases}$$

How many solutions does the equation $M(n) = 91$ have?

22. For each positive integer n, define $A_n = \dfrac{20^n + 11^n}{n!}$, where $n! = 1 \times 2 \times \cdots \times n$. Find the value of n that maximizes A_n.

23. Find the number of ways to pave a 1×10 block with tiles of sizes 1×1, 1×2 and 1×4, assuming tiles of the same size are indistinguishable. (For example, the following are two

distinct ways of using two tiles of size 1×1, two tiles of size 1×2 and one tile of size 1×4. It is not necessary to use all the three kinds of tiles.)



24. A 4×4 Sudoku grid is filled with digits so that each column, each row, and each of the four 2×2 sub-grids that composes the grid contains all of the digits from 1 to 4. For example,

4	3	1	2
2	1	3	4
1	2	4	3
3	4	2	1

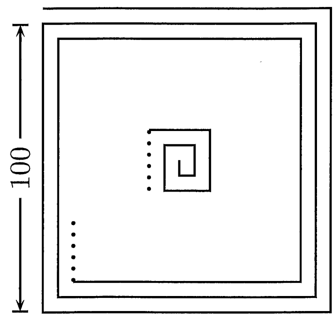
Find the total number of possible 4×4 Sudoku grids.

25. If the 13th of any particular month falls on a Friday, we call it *Friday the 13th*. It is known that Friday the 13th occurs at least once every calendar year. If the longest interval between two consecutive occurrences of Friday the 13th is x months, find x.

26. How many ways are there to put 7 identical apples into 4 identical packages so that each package has at least one apple?

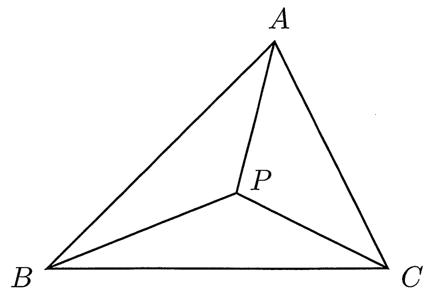
27. At a fun fair, coupons can be used to purchase food. Each coupon is worth \$5, \$8 or \$12. For example, for a \$15 purchase you can use three coupons of \$5, or use one coupon of \$5 and one coupon of \$8 and pay \$2 by cash. Suppose the prices in the fun fair are all whole dollars. What is the largest amount that you cannot purchase using only coupons?

28. Find the length of the spirangle in the following diagram, where the gap between adjacent parallel lines is 1 unit.



29. There are two fair dice and their sides are positive integers $a_1, \ldots, a_6$ and $b_1, \ldots, b_6$, respectively. After throwing them, the probability of getting a sum of $2, 3, 4, \ldots, 12$ respectively is the same as that of throwing two normal fair dice. Suppose that $a_1 + \cdots + a_6 < b_1 + \cdots + b_6$. What is $a_1 + \cdots + a_6$?

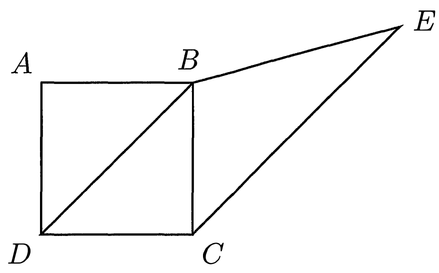
30. Consider a triangle ABC, where $AB = 20$, $BC = 25$ and $CA = 17$. P is a point on the plane. What is the minimum value of $2 \times PA + 3 \times PB + 5 \times PC$?



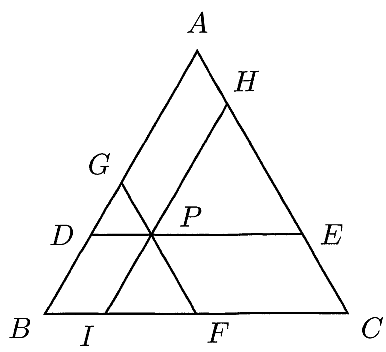
31. Given an equilateral triangle, what is the ratio of the area of its circumscribed circle to the area of its inscribed circle?

32. Let A and B be points that lie on the parabola $y = x^2$ such that both are at a distance of $8\sqrt{2}$ units from the line $y = -x - 4$. Find the square of the distance between A and B.

33. In the following diagram, $ABCD$ is a square, $BD \parallel CE$ and $BE = BD$. Let $\angle E = x^\circ$. Find x.



34. Consider an equilateral triangle ABC, where $AB = BC = CA = 2011$. Let P be a point inside $\triangle ABC$. Draw line segments passing through P such that $DE \parallel BC$, $FG \parallel CA$ and $HI \parallel AB$. Suppose $DE : FG : HI = 8 : 7 : 10$. Find $DE + FG + HI$.



35. In the following diagram, $AB \perp BC$. D and E are points on segments AB and BC respectively, such that $BA + AE = BD + DC$. It is known that $AD = 2$, $BE = 3$ and $EC = 4$. Find $BA + AE$.