Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Open Section, Round 2)

Saturday, 2 July 2011 **0900-1330**

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1. In the acute-angled non-isosceles triangle ABC, O is its circumcentre, H is its orthocentre and $AB > AC$. Let Q be a point on AC such that the extension of HQ meets the extension of BC at the point P. Suppose $BD = DP$, where D is the foot of the perpendicular from A onto BC. Prove that $\angle ODQ = 90°$.

2. If 46 squares are colored red in a 9×9 board, show that there is a 2×2 block on the board in which at least 3 of the squares are colored red.

3. Let $x, y, z > 0$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{1}{xyz}$. Show that

$$\frac{2x}{\sqrt{1+x^2}} + \frac{2y}{\sqrt{1+y^2}} + \frac{2z}{\sqrt{1+z^2}} < 3.$$

4. Find all polynomials $P(x)$ with real coefficients such that

$$P(a) \in \mathbb{Z} \quad \text{implies that} \quad a \in \mathbb{Z}.$$

5. Find all pairs of positive integers (m, n) such that

$$m + n - \frac{3mn}{m+n} = \frac{2011}{3}.$$