Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Senior Section Solutions)

1. Answer: (B)

   Since a, b and c are nonzero real numbers and $\dfrac{a^2}{b^2+c^2} < \dfrac{b^2}{c^2+a^2} < \dfrac{c^2}{a^2+b^2}$, we see that

   $$\frac{b^2+c^2}{a^2} > \frac{c^2+a^2}{b^2} > \frac{a^2+b^2}{c^2}.$$

   Adding 1 throughout, we obtain

   $$\frac{a^2+b^2+c^2}{a^2} > \frac{a^2+b^2+c^2}{b^2} > \frac{a^2+b^2+c^2}{c^2}.$$

   Thus $\dfrac{1}{a^2} > \dfrac{1}{b^2} > \dfrac{1}{c^2}$, which implies that $a^2 < b^2 < c^2$. So we have $|a| < |b| < |c|$.

2. Answer: (A)

   Note that

   $$(\sin\theta + \cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1 + \sin 2\theta = 1 + a.$$

   Since $0 \leq \theta \leq \frac{\pi}{2}$, we have $\sin\theta + \cos\theta > 0$. So $\sin\theta + \cos\theta = \sqrt{1+a}$.

3. Answer: (D)

   We have

   $$\begin{aligned} a^2+b^2+c^2-ab-bc-ca &= \frac{1}{2}\cdot[(a-b)^2+(b-e)^2+(c-a)^2] \\ &= \frac{1}{2}\cdot[(-1)^2+(-1)^2+2^2] = 3. \end{aligned}$$

4. Answer: (C)

   $$\begin{aligned} \frac{1}{x} - \frac{1}{2y} = \frac{1}{2x+y} \quad &\Rightarrow \quad \frac{2x+y}{x} - \frac{2x+y}{2y} = 1 \\ &\Rightarrow \quad 2 + \frac{y}{x} - \frac{x}{y} - \frac{1}{2} = 1 \\ &\Rightarrow \quad \frac{y}{x} - \frac{x}{y} = -\frac{1}{2}. \end{aligned}$$

   Now we have

   $$\frac{y^2}{x^2} + \frac{x^2}{y^2} = \left(\frac{y}{x} - \frac{x}{y}\right)^2 + 2 = \left(-\frac{1}{2}\right)^2 + 2 = \frac{9}{4}.$$

5. Answer: (E)

The area of $\triangle ABC$ is given to be $S = 1440\text{ cm}^2$. Let S_1 and S_2 denote the areas of $\triangle ADE$ and $\triangle DBE$ respectively. Since DE is parallel to AC, $\triangle DBE$ and $\triangle ABC$ are similar. Therefore

$$\frac{S_2}{S} = \left(\frac{BE}{BC}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}.$$

Thus $S_2 = \frac{9}{16}S$. As DE is parallel to AC, we have $AD : DB = CE : EB = 1 : 3$. Consequently,

$$\frac{S_1}{S_2} = \frac{1}{3}.$$

Hence $S_1 = \dfrac{1}{3}S_2 = \dfrac{1}{3}\cdot\dfrac{9}{16}S = \dfrac{3}{16}\times 1440 = 270\text{ cm}^2$.

6. Answer: (A)

Let $x = 2^{\frac{1}{4}}$. Then

$$\begin{aligned}
\frac{2}{\frac{1}{2^{\frac{1}{2}}+2^{\frac{3}{4}}+2}+\frac{1}{2^{\frac{1}{2}}-2^{\frac{3}{4}}+2}} &= \frac{2}{\frac{1}{x^2+x^3+2}+\frac{1}{x^2-x^3+2}} \\
&= \frac{2}{\frac{2(x^2+2)}{(x^2+2)^2-x^6}} \\
&= \frac{(x^2+2)^2-x^6}{x^2+2} \\
&= \frac{(\sqrt{2}+2)^2-2\sqrt{2}}{\sqrt{2}+2} \\
&= \frac{6+2\sqrt{2}}{\sqrt{2}+2}\times\frac{2-\sqrt{2}}{2-\sqrt{2}} \\
&= \frac{8-2\sqrt{2}}{2} \\
&= 4-\sqrt{2}.
\end{aligned}$$

7. Answer: (E)

$$\begin{aligned}
x &= \frac{\log(\frac{1}{3})}{\log(\frac{1}{2})}+\frac{\log(\frac{1}{5})}{\log(\frac{1}{4})}+\frac{\log(\frac{1}{7})}{\log(\frac{1}{8})} = \frac{-\log 3}{-\log 2}+\frac{-\log 5}{-\log 4}+\frac{-\log 7}{-\log 8} \\
&= \frac{\log 3+\log 5^{\frac{1}{2}}+\log 7^{\frac{1}{3}}}{\log 2} \\
&= \frac{\log\sqrt{45}+\log 7^{\frac{1}{3}}}{\log 2} < \frac{\log\sqrt{64}+\log 8^{\frac{1}{3}}}{\log 2} = \frac{3\log 2+\log 2}{\log 2} = 4.
\end{aligned}$$

Moreover,

$$\begin{aligned} 2x &= \frac{2(\log 3 + \log 5^{\frac{1}{2}} + \log 7^{\frac{1}{3}})}{\log 2} \\ &= \frac{\log(9 \times 5) + \log(49^{\frac{1}{3}})}{\log 2} > \frac{\log(45 \times 27^{\frac{1}{3}})}{\log 2} \\ &= \frac{\log(45 \times 3)}{\log 2} > \frac{\log(128)}{\log 2} = 7, \end{aligned}$$

so $x > 3.5$.

8. Answer: (B)

Note that $7^4 - 1 = 2400$, so that $7^{4n} - 1$ is divisible by 100 for any $n \in \mathbb{Z}^+$. Now,

$$\begin{aligned} 7^{5^6} &= 7(7^{5^6-1} - 1 + 1) \\ &= 7(7^{5^6-1} - 1) + 7 \\ &= 7(7^{4n} - 1) + 7, \end{aligned}$$

where

$$n = \frac{5^6 - 1}{4} \in \mathbb{Z}^+.$$

Since $7(7^{4n} - 1)$ is divisible by 100, its last two digits are 00. It follows that the last two digits of 7^{5^6} are 07.

9. Answer: (A)

$$\begin{aligned} \frac{\log_x 2011 + \log_y 2011}{\log_{xy} 2011} &= (\frac{\log 2011}{\log x} + \frac{\log 2011}{\log y}) \cdot (\frac{\log xy}{\log 2011}) \\ &= (\frac{1}{\log x} + \frac{1}{\log y}) \cdot (\log x + \log y) \\ &= 2 + \frac{\log x}{\log y} + \frac{\log y}{\log x} \\ &\geq 4 \text{ (using } AM \geq GM), \end{aligned}$$

and the equality is attained when $\log x = \log y$, or equivalently, $x = y$.

10. Answer: (C)

The roots of the equation $x^2 - (c-1)x + c^2 - 7c + 14 = 0$ are a and b, which are real. Thus the discriminant of the equation is non-negative. In other words,

$$(c-1)^2 - 4(c^2 - 7c + 14) = -3c^2 + 26c - 55 = (-3c + 11)(c - 5) \geq 0.$$

So we have $\frac{11}{3} \leq c \leq 5$. Together with the equalities

$$\begin{aligned} a^2 + b^2 &= (a+b)^2 - 2ab \\ &= (c-1)^2 - 2(c^2 - 7c + 14) \\ &= -c^2 + 12c - 27 = 9 - (c-6)^2, \end{aligned}$$

we see that maximum value of $a^2 + b^2$ is 8 when $c = 5$.

11. Answer: 400

$$\frac{2011^2 + 2111^2 - 2 \times 2011 \times 2111}{25} = \frac{(2011 - 2111)^2}{25} = \frac{100^2}{25} = 400.$$

12. Answer: 12

Since $6033 = 3 \times 2011$, we have

$$\begin{aligned} n^{6033} < 2011^{2011} &\iff n^{3\times 2011} < 2011^{2011} \\ &\iff n^3 < 2011. \end{aligned}$$

Note that $12^3 = 1728$ and $13^3 = 2197 > 2011$. Thus, the largest possible natural number n satisfying the given inequality is 12.

13. Answer: 14

$$(1 + \sqrt{3})^2 = 1^2 + 3 + 2\sqrt{3} = 4 + 2\sqrt{3}.$$

$$\begin{aligned} (1 + \sqrt{3})^4 = ((1 + \sqrt{3})^2)^2 = (4 + 2\sqrt{3})^2 &= 4^2 + (2\sqrt{3})^2 + 2 \cdot 4 \cdot 2\sqrt{3} \\ &= 16 + 12 + 16\sqrt{3} \\ &= 28 + 16\sqrt{3}. \end{aligned}$$

Hence,

$$\frac{(1 + \sqrt{3})^4}{4} = \frac{28 + 16\sqrt{3}}{4} = 7 + 4\sqrt{3}.$$

Note that $1.7 < \sqrt{3} < 1.8$. Thus,

$$\begin{aligned} & 7 + 4 \times 1.7 < 7 + 4\sqrt{3} < 7 + 4 \times 1.8 \\ \Longrightarrow \quad & 13.8 < 7 + 4\sqrt{3} < 14.2. \end{aligned}$$

Therefore, the integer closest to $\frac{(1+\sqrt{3})^4}{4}$ is 14.

14. Answer: 300

Let P be the intersection of CM and BN, so that P is the centroid of $\triangle ABC$. Then $BP = 2PN$ and $CP = 2PM$. Let $PN = PM = y$, so that $BP = CP = 2y$. Since $\angle BPC = 90°$, we have $BC = 20 = \sqrt{(2y)^2 + (2y)^2}$ and thus $y = \sqrt{50}$. Now

$$AB^2 = 4BM^2 = 4((2y)^2 + y^2) = 20y^2 = 1000.$$

Thus the altitude of $\triangle ABC$ is $\sqrt{AB^2 - 10^2} = 30$. Hence the area of $\triangle ABC$ is $\frac{1}{2} \times 20 \times 30 = 300$.

15. Answer: 8001

Note that $\sqrt{5n}-\sqrt{5n-4}<0.01$ if and only if

$$\sqrt{5n}+\sqrt{5n-4}=\frac{4}{\sqrt{5n}-\sqrt{5n-4}}>400.$$

If $n=8000$, then $\sqrt{5n}+\sqrt{5n-4}=\sqrt{40000}+\sqrt{39996}<400$.

If $n=8001$, then $\sqrt{5n}+\sqrt{5n-4}=\sqrt{40005}+\sqrt{40001}>400$.

So the answer is 8001.

16. Answer: 1006

The series can be paired as

$$(\frac{1}{1+11^{-2011}}+\frac{1}{1+11^{2011}})+(\frac{1}{1+11^{-2009}}+\frac{1}{1+11^{2009}})+\cdots+(\frac{1}{1+11^{-1}}+\frac{1}{1+11^{1}}).$$

Each pair of terms is of the form

$$\frac{1}{1+a^{-1}}+\frac{1}{1+a}=1.$$

There are 1006 pairs of such terms, and thus the sum of the series is 1006.

17. Answer: 54

$$\begin{aligned} x &= \sin^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{7\pi}{8}\right)+\cos^4\left(\frac{7\pi}{8}\right) \\ &= \sin^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right) \\ &= 2(\sin^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{\pi}{8}\right))^2-4\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)=2-\sin^2\left(\frac{\pi}{4}\right)=\frac{3}{2}. \end{aligned}$$

Thus $36x=54$.

18. Answer: 1005

By the laws of sine and cosine, we have

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} \quad \text{and} \quad \cos C=\frac{a^2+b^2-c^2}{2ab}.$$

Then

$$\begin{aligned} \frac{\cot C}{\cot A+\cot B} &= \frac{\cos C}{\sin C}\cdot\frac{1}{\frac{\cos A\sin B+\cos B\sin A}{\sin A\sin B}} \\ &= \frac{\sin A\sin B\cos C}{\sin(A+B)\sin C} \\ &= (\frac{\sin A\sin B}{\sin^2 C})\cos C \\ &= (\frac{(ab/c^2)\sin^2 C}{\sin^2 C})(\frac{a^2+b^2-c^2}{2ab}) \\ &= \frac{a^2+b^2-c^2}{2c^2} \\ &= \frac{2011-1}{2}=1005. \end{aligned}$$

19. Answer: 1011

It is given that there are at least 1000 female participants. Suppose there are more than 1000 female participants. Then we take a group of 1001 female participants, and add any 10 participants to this group of female participants. This will result in a group of 1011 participants with at most 10 male participants, which contradicts the assumption. Therefore, there are exactly 1000 female participants. Hence, the number of male participants is $2011 - 1000 = 1011$.

20. Answer: 1

Substituting $x = \frac{1}{2}, -1, 2$, we get

$$\begin{array}{rcrcrcl} \frac{1}{4}f(\frac{1}{2}) & + & f(-1) & & & = & \frac{1}{2}, \\ & & f(-1) & + & f(2) & = & 2, \\ f(\frac{1}{2}) & + & & & 4f(2) & = & 8. \end{array}$$

Solving these equations, we get $f(\frac{1}{2}) = 1$. In fact the same method can be used to determine f. Letting $x = z, \frac{z-1}{z}, \frac{1}{1-z}$, we get

$$\begin{array}{rcrcrcl} z^2 f(z) & + & f\left(\frac{z-1}{z}\right) & & & = & 2z^2, \\ & & \left(\frac{z-1}{z}\right)^2 f\left(\frac{z-1}{z}\right) & + & f\left(\frac{1}{1-z}\right) & = & 2\left(\frac{z-1}{z}\right)^2, \\ f(z) & + & & & \frac{1}{(1-z)^2} f\left(\frac{1}{1-z}\right) & = & \frac{1}{2(1-z)^2}. \end{array}$$

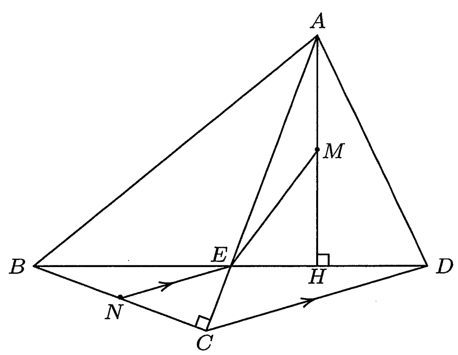
Using Cramer's rule, we can solve this system of linear equations in the unknowns $f(z), f\left(\frac{z-1}{z}\right), f\left(\frac{1}{1-z}\right)$. We obtain

$$f(z) = 1 + \frac{1}{(1-z)^2} - \frac{1}{z^2}.$$

Indeed one can easily check that it satisfies the given functional equation.

21. Answer: 40

Let M be the midpoint of AH and N the midpoint of BC. Then CD is parallel to NE and $CD = 2NE$. Since $\triangle AHE$ is similar to $\triangle BCE$, we have $\triangle MHE$ is similar to $\triangle NCE$. As $ME = \sqrt{15^2 + 20^2} = 25$, we thus have $NE = \frac{EC}{EH} \cdot ME = \frac{12 \times 25}{15} = 20$. Therefore $CD = 2NE = 40$.



22. Answer: 3

Note that

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{211}\Rightarrow xy-211x-211y=0\Rightarrow(x-211)(y-211)=211^2.$$

Since 211 is a prime number, the factors of 211^2 are 1, 211, 211^2, -1, -211, -211^2. Thus the pairs of integers (x,y) satisfying the last equation are given by:

$$(x-211,\ y-211)=(1,\ 211^2),\ (211,\ 211),\ (211^2,\ 1),\ (-1,\ -211^2),$$
$$(-211,\ -211),\ (-211^2,\ -1).$$

Equivalently, (x,y) are given by

$$(212,\ 211+211^2),\ (422,\ 422),\ (211+211^2,\ 212),\ (210,\ 211-211^2),$$
$$(0,\ 0),\ (211-211^2,\ 210).$$

Note that (0, 0) does not satisfy the first equation. Among the remaining 5 pairs which satisfy the first equation, three of them satisfy the inequality $x\geq y$, and they are given by $(x,\ y)=(422,\ 422),\ (211+211^2,\ 212),\ (210,\ 211-211^2)$.

23. Answer: 93

By long division, we have

$$x^4+ax^2+bx+c=(x^2+3x-1)\cdot(x^2-3x+(a+10))+(b-3a-33)x+(c+a+10).$$

Let m_1,m_2 be the two roots of the equation $x^2+3x-1=0$. Note that $m_1\neq m_2$, since the discriminant of the above quadratic equation is $3^2-4\cdot1\cdot1\cdot(-1)=13\neq0$. Since m_1, m_2 also satisfy the equation $x^4+ax^2+bx+c=0$, it follows that m_1 and m_2 also satisfy the equation

$$(b-3a-33)x+(c+a+10)=0.$$

Thus we have

$$(b-3a-33)m_1+(c+a+10)=0,$$

and

$$(b-3a-33)m_2+(c+a+10)=0.$$

Since $m_1\neq m_2$, it follows that $b-3a-33=0$ and $c+a+10=0$. Hence we have $b=3a+33$ and $c=-a-10$. Thus $a+b+4c+100=a+(3a+33)+4(-a-10)+100=93$.

24. Answer: 1120

Let m and n be positive integers satisfying the given equation. Then $3(n^2-m)=2011n$. Since 2011 is a prime, 3 divides n. By letting $n=3k$, we have $(3k)^2=m+2011k$. This implies that k divides m. Let $m=rk$. Then $9k^2=rk+2011k$ so that $9k=r+2011$. The smallest positive integer r such that $r+2011$ is divisible by 9 is $r=5$. Thus $k=(5+2011)/9=224$. The corresponding values of m and n are $m=1120$ and $n=672$.

25. Answer: 1

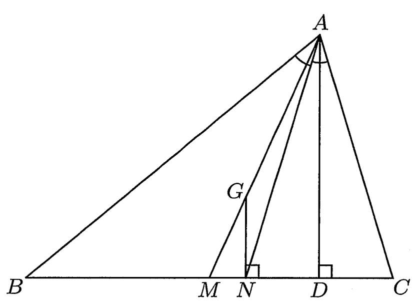
We shall show that the only possible values of a, b, c are $a = b = c = 1$ so that $abc = 1$. From the first equation, we note that $a^2 - b = (x - a)^2$ is a perfect square less than a^2. Thus $a^2 - b \leq (a-1)^2$. That is $b \geq 2a - 1$. Likewise $c \geq 2b - 1$ and $a \geq 2c - 1$. Combining these inequalities, we have $a \geq 8a - 7$ or $a \leq 1$. Thus $a = 1$. Similarly $b = c = 1$.

26. Answer: 180

Note that
$$A = \frac{\pi}{2} - B - C \leq \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$
and $\sin(B - C) \geq 0$. Therefore
$$\begin{aligned}\sin A \cos B \sin C &= \frac{1}{2} \cdot \sin A \cdot [\sin(B + C) - \sin(B - C)] \\ &\leq \frac{1}{2} \sin A \sin(B + C) \\ &= \frac{1}{2} \sin A \cos A \\ &= \frac{1}{4} \sin 2A \leq \frac{1}{4} \sin \frac{\pi}{2} = \frac{1}{4}.\end{aligned}$$
When $A = \frac{\pi}{4}$ and $B = C = \frac{\pi}{8}$, we have
$$\sin A \cos B \sin C = \sin \frac{\pi}{4} \cos \frac{\pi}{8} \sin \frac{\pi}{8} = \sin \frac{\pi}{4} \cdot \frac{1}{2} \sin \frac{\pi}{4} = \frac{1}{4}.$$
Hence the largest possible value of $720(\sin A)(\cos B)(\sin C)$ is $720 \times \frac{1}{4} = 180$.

27. Answer: 9

Let $BC = a, CA = b$ and $AB = c$. We shall prove that $c + b = a\sqrt{3}$. Thus $c = 5\sqrt{3} \times \sqrt{3} - 6 = 9$. Using the angle bisector theorem, we have $BN/NC = c/a > 1$ so that $BN > BM$. Also $\angle ANC = \angle B + \frac{1}{2}\angle A < \angle C + \frac{1}{2}\angle A < \angle ANB$ so that $\angle ANC$ is acute. This shows that B, M, N, D are in this order, where M is the midpoint of BC and D is the foot of the perpendicular from A onto BC.



First we have $BD = c \cos B = \frac{a^2+c^2-b^2}{2a}$. Using the angle bisector theorem, $BN = \frac{ac}{b+c}$. Therefore, $MN = BN - \frac{a}{2} = \frac{ac}{b+c} - \frac{a}{2} = \frac{a(c-b)}{2(c+b)}$. Also $MD = BD - \frac{a}{2} = \frac{a^2+c^2-b^2}{2a} -$

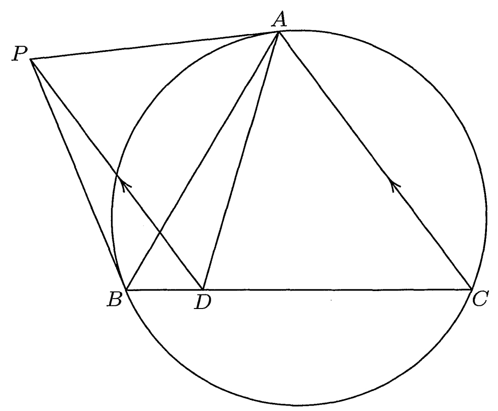
$\frac{a}{2} = \frac{c^2-b^2}{2a}$. Since GN is parallel to AD and G is the centroid of the triangle ABC, we have $MD/MN = 3$. It follows that $c+b = a\sqrt{3}$. Thus, $AB = a\sqrt{3} - b = 15 - 6 = 9$.

28. Answer: 28

The sum of 55 positive consecutive integers is at least $(55 \times 56)/2 = 1540$. Let the middle number of these consecutive positive integers be x. Then the product $abcd = 55x = 5 \cdot 11 \cdot x$. So we have $55x \geq 1540$ and thus $x \geq 28$. The least value of $a + b + c + d$ is attained when $x = 5(7)$. Thus the answer is $5 + 11 + 5 + 7 = 28$.

29. Answer: 168

First $\angle BDP = \angle BCA = \angle BAP$ so that P, B, D, A are concyclic. Thus $\angle ACD = \angle PBA = \angle PDA = \angle DAC$ so that $DA = DC$.



By cosine rule, $\cos C = 3/5$. Thus $DC = \frac{1}{2}AC/\cos C = 21 \times 5/3 = 35$. Hence $BD = 10$ and $BC = 10 + 35 = 45$. Thus area$(\triangle ABD) = \frac{10}{45} \times$ area$(\triangle ABC)$. By Heron's formula, area$(\triangle ABC) = 756$. Thus area$(\triangle ABD) = \frac{10}{45} \times 756 = 168$.

30. Answer: 276

Since the number of positive divisors of a is odd, a must be a perfect square. As a is a divisor of $4400 = 2^4 \times 5^2 \times 11$ and a has exactly 9 positive divisors, we see that $a = 2^2 \times 5^2$. Now the least common multiple of a and b is 4400 implies that b must have $2^4 \times 11$ as a divisor. Since $2^4 \times 11$ has exactly 10 positive divisors, we deduce that $b = 2^4 \times 11 = 176$. Hence $a + b = 276$.

31. Answer: 20

First we let ℓ be the line which extends BC in both directions. Let E be the point on ℓ such that AE is perpendicular to ℓ. Similarly, we let F be the point on ℓ such that DF is perpendicular to ℓ. Then, it is easy to see that $BE = AE = \sqrt{6}$, $CF = 2\sqrt{2}$ and $DF = 2\sqrt{6}$. Thus $EF = \sqrt{6} + 4 - 2\sqrt{2} + 2\sqrt{2} = 4 + \sqrt{6}$. Now we let G be the point on DF such that AG is parallel to ℓ. Then $AG = EF = 4 + \sqrt{6}$ and

$DG = DF - GF = DF - AE = 2\sqrt{6} - \sqrt{6} = \sqrt{6}$. So for the right-angled triangle ADG, we have

$$x = AD = \sqrt{AG^2 + DG^2} = \sqrt{(4+\sqrt{6})^2 + (\sqrt{6})^2} = \sqrt{28 + 8\sqrt{6}} = 2 + 2\sqrt{6}.$$

Thus, $x^2 - 4x = (2+2\sqrt{6})^2 - 8 - 8\sqrt{6} = 4 + 24 + 8\sqrt{6} - 8 - 8\sqrt{6} = 20$.

32. Answer: 5

    Suppose x and y are positive integers satisfying $x^3 + y^3 - 3xy = p - 1$. That is $(x+y+1)(x^2+y^2-xy-x-y+1) = p$. Since $x, y \geq 1$, we must have $x+y+1 = p$ and $x^2+y^2-xy-x-y+1 = 1$. Suppose $x = y$. Then the second equation gives $x = y = 2$. Thus $p = x+y+1 = 5$. Next we may suppose without loss of generality that $x > y \geq 1$. Thus $x - y \geq 1$. Then the equation $x^2+y^2-xy-x-y+1 = 1$ can be written as $x + y - xy = (x-y)^2 \geq 1$. That is $(x-1)(y-1) \leq 0$. It follows that $x = 1$ or $y = 1$. Since we assume $x > y$, we must have $y = 1$. Then from $x^2+y^2-xy-x-y+1 = 1$, we get $x = 2$. But then $x+y+1 = 4$ is not a prime. Consequently there is no solution in x and y if $x \neq y$. Therefore the only solution is $x = y = 2$ and $p = 5$.

33. Answer: 61

    Without loss of generality, we may assume that $a \geq b \geq c$. Let the HCF (or GCD) of a, b and c be d. Then $a = da_1$, $b = db_1$ and $c = dc_1$. Let the LCM of a_1, b_1 and c_1 be m. Thus, $a_1^2 + b_1^2 + c_1^2 = \frac{2011}{d^2}$ and $d + md = 388$ or $1 + m = \frac{388}{d}$. So, $d^2 \mid 2011$ and $d \mid 388$. Note that 2011 is a prime. Thus we must have $d = 1$, and it follows that $a = a_1$, $b = b_1$, $c = c_1$, and thus $a^2 + b^2 + c^2 = 2011$. In particular, $a^2 + b^2 + c^2 < 2025 = 45^2$, so that one has $a, b, c < 45$. Furthermore we have $m = 387 = 3^2 \times 43$. Thus a, b and c can only be 1, 3, 9 or 43, since they must be less than 45. Then it is easy to check that $43^2 + 9^2 + 9^2 = 2011$, and $a = 43$, $b = c = 9$ is the only combination which satisfies the given conditions. Thus we have $a + b + c = 43 + 18 = 61$.

34. Answer: 16

    Consider the subset $T = \{1, 2^2, 3^2, 5^2, 7^2, \cdots, 43^2\}$ consisting of the number 1 and the squares of all prime numbers up to 43. Then $T \subseteq S$, $|T| = 15$, and all elements in T are pairwise relatively prime; however, T contains no prime number. Thus $k \geq 16$. Next we show that if A is any subset of S with $|A| = 16$ such that all elements in A are pairwise relatively prime, then A contains a prime number. Suppose to the contrary that A does not contain a prime number. Let $A = \{a_1, a_2, \cdots, a_{16}\}$. We consider two cases:

    Case 1. 1 is not in A. Then $a_1, a_2, \cdots, a_{16}$ are composite numbers. Let p_i be the smallest prime factor of a_i, $i = 1, 2, \cdots, 16$. Since $\gcd(a_i, a_j) = 1$ for $i \neq j$, $i, j = 1, 2, \cdots, 16$, we see that the prime numbers $p_1, p_2, \cdots, p_{16}$ are all distinct. By re-ordering $a_1, a_2, \cdots, a_{16}$ if necessary, we may assume that $p_1 < p_2 < \cdots < p_{16}$. In addition, since each a_i is a composite number, it follows that

    $$a_1 \geq p_1^2 \geq 2^2,\ a_2 \geq p_2^2 \geq 3^2, \cdots, a_{15} \geq p_{15}^2 \geq 47^2 > 2011,$$

which is a contradiction to the given fact that each element of S is less than or equal to 2011.

Case 2. 1 is in A. We may let $a_{16} = 1$. Then $a_1, a_2, \cdots, a_{15}$ are composite numbers. As in Case 1, we have

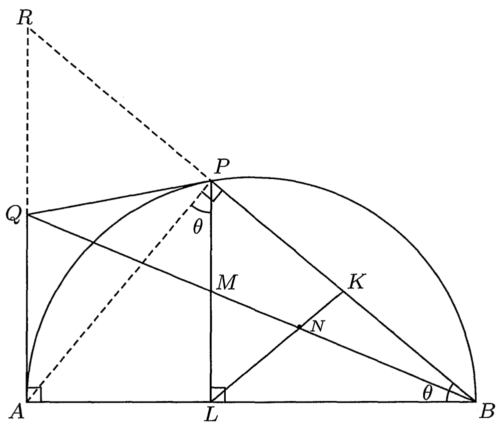
$$a_1 \geq p_1^2 \geq 2^2,\ a_2 \geq p_2^2 \geq 3^2, \cdots,\ a_{15} \geq p_{15}^2 \geq 47^2 > 2011,$$

which is a contradiction.

Thus we have shown that every 16-element subset A of S such that all elements in A are pairwise relatively prime must contain a prime number. Hence the smallest k is 16.

35. Answer: 12

Let the extensions of AQ and BP meet at the point R. Then $\angle PRQ = \angle PAB = \angle QPR$ so that $QP = QR$. Since $QA = QP$, the point Q is the midpoint of AR. As AR is parallel to LP, the triangles ARB and LPB are similar so that M is the midpoint of PL. Therefore, N is the centroid of the triangle PLB, and $3MN = BM$.



Let $\angle ABP = \theta$. Thus $\tan\theta = AR/AB = 2AQ/AB = 5/6$. Then $BL = PB\cos\theta = AB\cos^2\theta$. Also $BM/BL = BQ/BA$ so that $3MN = BM = \frac{BQ}{AB}AB\cos^2\theta = \cos^2\theta(QM + 3MN)$. Solving for MN, we have $MN = \frac{QM}{3\tan^2\theta} = \frac{25}{3\times(5/6)^2} = 12$.