Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Junior Section)

Multiple Choice Questions

1. Answer: (D)

   Note that $\frac{n}{2^n} = \frac{n+1}{2^{n-1}} - \frac{n+2}{2^n}$. Then

$$\begin{aligned}\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{10}{2^{10}} &= \left(\frac{2}{2^0}-\frac{3}{2^1}\right)+\left(\frac{3}{2^1}-\frac{4}{2^2}\right)+\left(\frac{4}{2^2}-\frac{5}{2^3}\right)+\cdots+\left(\frac{11}{2^9}-\frac{12}{2^{10}}\right)\\ &= \frac{2}{2^0}-\frac{12}{2^{10}}=\frac{509}{256}.\end{aligned}$$

2. Answer: (C).

   Let the roots be $n_1 \le n_2 \le n_3 \le n_4 \le n_5$. Then the polynomial can be factorized as

   $(x-n_1)(x-n_2)(x-n_3)(x-n_4)(x-n_5) = x^5-(n_1+n_2+n_3+n_4+n_5)x^4+\cdots-n_1n_2n_3n_4n_5.$

   Compare the coefficients: $n_1+n_2+n_3+n_4+n_5=-3$ and $n_1n_2n_3n_4n_5 = 2011^2$. Then

$$n_1=-2011, \quad n_2=n_3=n_4=-1, \quad n_5=2011.$$

3. Answer: (C).

   If $z=2$, then $(x,y)=(1,1)$.

   If $z=3$, then $(x,y)=(1,2),(2,1)$.

   If $z=4$, then $(x,y)=(1,3),(2,2),(3,1)$.

   If $z=5$, then $(x,y)=(1,4),(2,3),(3,2),(4,1)$.

   If $z=6$, then $(x,y)=(1,5),(2,4),(3,3),(4,2),(5,1)$.

   Out if these 15 cases, 8 of them contain at least one 2. Hence, the required probability is $\frac{8}{15}$.

4. Answer: (B).

Note that

$$x = (10^{1012} + 1) \times 10^{2014} + 50 = 10^{4026} + 10^{2014} + 50$$
$$= (10^{2013})^2 + 2 \times 10^{2013} \times 5 + 50 = (10^{2013} + 5)^2 + 25.$$

So $x - 25$ is a perfect square.

5. Answer: (B).

Let $K = N_2 + \cdots + N_{2010}$. Then $X = (N_1 + K)(K + N_{2011})$ and $Y = (N_1 + K + N_{2011})K$.

$$X - Y = (N_1K + K^2 + N_1N_{2011} + KN_{2011}) - (N_1K + K^2 + N_{2011}K) = N_1N_{2011} > 0.$$

6. Answer: (A).

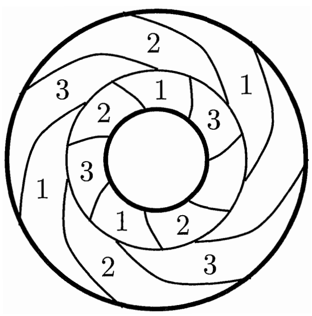
The area enclosed by AD, DE and $\overset{\frown}{AE}$ is $\dfrac{\pi(2^2)}{8} - 1 = \dfrac{\pi}{2} - 1$.

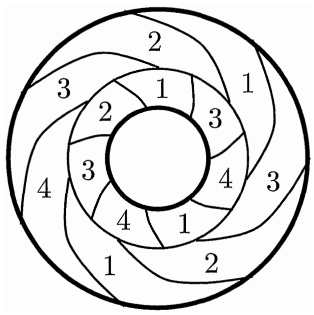
The area of the wedge EDF is $\dfrac{\pi(2-\sqrt{2})^2}{4} = \left(\dfrac{3}{2} - \sqrt{2}\right)\pi$.

So the area of the egg is: $\dfrac{\pi}{2} + 1 + 2 \times \left(\dfrac{\pi}{2} - 1\right) + \left(\dfrac{3}{2} - \sqrt{2}\right)\pi = (3 - \sqrt{2})\pi - 1.$

7. Answer: (B).

The left shows that 3 colours are not enough. The right is a painting using 4 colours.





8. Answer: (E).

Since $5 \mid (2^4 - 1)$, $7 \mid (3^6 - 1)$, $11 \mid (5^{10} - 1)$, $13 \mid (7^{12} - 1)$, n is divisible by $5, 7, 11$ and 13.

9. Answer: (C).

We consider the position of the Black Knight. The number of positions being attacked by the White Knight can be counted.

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

There are $16 \times 8 + 16 \times 6 + 20 \times 4 + 8 \times 3 + 4 \times 2 = 336$ cases. Hence, the total number of cases that the Kights do not attack each other is $64 \times 63 - 336 = 3696$.

10. Answer: (E).

Note that the power of any positive integer n with last digit 1 or 6 is 1 or 6 respectively.

If the last digit of n is 9, then $n^2 \equiv 1 \pmod{10}$, and $n^n \equiv n^{10k+9} \equiv -1 \equiv 9 \pmod{10}$.

If the last digit of n is 7, then $n^4 \equiv 1 \pmod{10}$. Suppose the second last digit of n is odd. Then $n^n \equiv n^{20k+17} \equiv 7 \pmod{10}$.

Short Questions

11. Answer: 2.

$$2 = \left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+2\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ca}\right) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}+2\,\frac{xyz}{abc}\left(\frac{c}{z}+\frac{a}{x}+\frac{b}{y}\right).$$

So $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} = 2$.

12. Answer: 5.

Note that $x = \dfrac{1}{13\sqrt{(4+\sqrt{3})^2}} = \dfrac{13(4-\sqrt{3})}{(4-\sqrt{3})(4+\sqrt{3})} = 4-\sqrt{3}$. So $(x-4)^2 = 3$. That is,

$$x^2 - 8x + 15 = 2.$$

It follows that

$$\begin{aligned}\frac{x^4-6x^3-2x^2+18x+23}{x^2-8x+15} &= x^2+2x-1+\frac{38-20x}{x^2-8x+15}\\ &= x^2+2x-1+\frac{38-20x}{2}\\ &= x^2-8x+18\\ &= 2+3=5.\end{aligned}$$

13. Answer: 3.

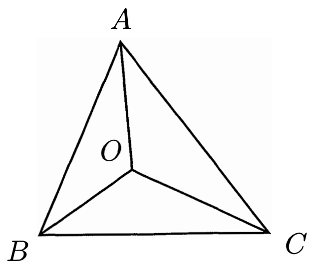
Let $f(x) = \dfrac{\sqrt{3}x-1}{x+\sqrt{3}}$. Then $f(f(x)) = \dfrac{\sqrt{3}\frac{\sqrt{3}-1}{x+\sqrt{3}}-1}{\frac{\sqrt{3}x-1}{x+\sqrt{3}}+\sqrt{3}} = \dfrac{3x-\sqrt{3}-x-\sqrt{3}}{\sqrt{3}x-1+\sqrt{3}x+3} = \dfrac{x-\sqrt{3}}{\sqrt{3}x+1}$.

$f(f(f(x))) = \dfrac{\frac{\sqrt{3}x-1}{x+\sqrt{3}}-\sqrt{3}}{\sqrt{3}\frac{\sqrt{3}x-1}{x+\sqrt{3}}+1} = \dfrac{\sqrt{3}x-1-\sqrt{3}x-3}{3x-\sqrt{3}+x+\sqrt{3}} = -\dfrac{1}{x}$. So $f(f(f(f(f(f(x)))))) = x$.

Since $2010 = 6 \times 335$, $a_{2011} = \underbrace{f(f(f\cdots f(f}_{2010 \text{ copies}}(3))\cdots)) = 3$.

14. Answer: 129.

Consider the following picture, where $\angle AOB = \angle BOC = \angle COA = 120°$, $OA = a$, $OB = b$ and $OC = c$.



Then $|BC| = 5$, $|CA| = 7$ and $|AB| = 8$. The area of the triangle ABC is

$$\sqrt{10(10-5)(10-7)(10-8)} = 10\sqrt{3}.$$

Then $\frac{1}{2}\frac{\sqrt{3}}{2}(ab+bc+ca) = 10\sqrt{3}$. So $ab+bc+ca = 40$.

$$2(a+b+c)^2 = (a^2+ab+b^2)+(b^2+bc+c^2)+(c^2+ca+a^2)+3(ab+bc+ca) = 258.$$

Thus, $(a+b+c)^2 = 129$.

15. Answer: 0.

Define $Q(x) = (1+x)P(x) - x$. Then $Q(x)$ is a polynomial of degree 2011. Since $Q(0) = Q(1) = Q(2) = \cdots = Q(2010) = 0$, we can write, for some constant A,

$$Q(x) = Ax(x-1)(x-2)\cdots(x-2010).$$

$1 = Q(-1) = A(-1)(-2)(-3)\cdots(-2011) = -A \cdot 2011!$. Then $Q(2012) = A \cdot 2012! = -2012$, and $P(2012) = \dfrac{Q(2012)+2012}{2013} = 0$.

16. Answer: 9241.

Let $n = \lfloor x \rfloor$, $\{x\} = x - n$. The equation becomes $(n + \{x\})^3 - \{x\}^3 = \lfloor (n + \{x\})^3 \rfloor$. Then

$$3n\{x\}(n + \{x\}) = \left\lfloor 3n\{x\}(n + \{x\}) + \{x\}^3 \right\rfloor.$$

The right-hand side is an integer. The above holds if and only if $3n\{x\}(n+\{x\})$ is an integer.

Note that $0 \leq 3n\{x\}(n+\{x\}) < 3n(n+1)$. There are exactly $3n(n+1)$ solutions in $[n, n+1)$, $n = 1, 2, \ldots$. So on $[1, 20]$, the total number of solutions is

$$\begin{aligned} & 3(1 \times 2 + 2 \times 3 + \cdots 20 \times 21) + 1 \\ = & (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (21^3 - 20^3) - 20 + 1 \\ = & 21^3 - 20 = 9241. \end{aligned}$$

17. Answer: 224.

For smallest possible n, we need to have 9 as the digits of n as many as possible. So n is the integer whose first digit is $2011 - 223 \times 9 = 4$ and followed by 223 9's.

18. Answer: 1001.

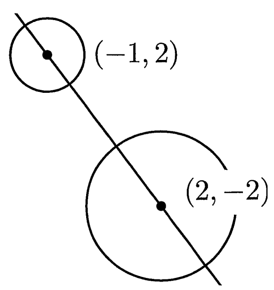
$\dfrac{n^3 + 2011}{n + 10} = n^2 - 10n + 100 + \dfrac{1011}{n + 10}$. This is an integer if and only if $(n + 10) \mid 1011$. The maximum value of n is $1011 - 10 = 1001$.

19. Answer: 16.

Complete the squares: $(a + 1)^2 + (b - 2)^2 = 1$ and $(c - 2)^2 + (d + 2)^2 = 2^2$. Each of them represents a circle.

The distance between the two centres is $\sqrt{(2 - (-1))^2 + (-2 - 2)^2} = 5$.

So $m = 5 - 1 - 2 = 2$ and $M = 5 + 1 + 2 = 8$. Thus, $m \times M = 16$.



20. Answer: 4022.

Suppose $x_1 = x_2 = \cdots = x_k = 1 < 2 \le x_{k+1} \le \cdots \le x_{2011}$. Let $M = x_1 \cdots x_{2011}$. Then

$$\begin{aligned} M &= x_{k+1}x_{k+2}\cdots x_{2010}x_{2011} \\ &= (x_{k+1}-1)x_{k+2}\cdots x_{2010}x_{2011} + x_{k+2}\cdots x_{2010}x_{2011} \\ &\ge (x_{k+1}-1)2 + x_{k+2}\cdots x_{2010}x_{2011} \\ &\ge \cdots\cdots\cdots\cdots \\ &\ge (x_{k+1}-1)2 + \cdots + (x_{2009}-1)2 + x_{2010}x_{2011} \\ &\ge (x_{k+1}-1)2 + \cdots + (x_{2009}-1)2 + (x_{2010}-1)2 + (x_{2011}-1)2 \\ &= 2(x_{k+1} + \cdots + x_{2011} - (2011-k)) \\ &= 2(M - 2011). \end{aligned}$$

Therefore, $M \le 4022$. On the other hand, $(1, 1, \ldots, 1, 2, 2011)$ is a solution to the equation. So the maximum value is 4022.

21. Answer: 101.

If $n \ge 102$, then $M(n) = n - 10 \ge 92$.

$M(91) = M(M(102)) = M(92) = M(M(103)) = M(93) = \cdots = M(101) = 91$.

For each $k = 1, \ldots, 10$, $M(80+k) = M(M(91+k)) = M(91) = 91$, and thus

$$\begin{gathered} M(70+k) = M(M(81+k)) = M(91) = 91, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ M(k) = M(M(11+k)) = M(91) = 91. \end{gathered}$$

Hence, all integers from 1 to 101 are solutions to $M(n) = 91$.

22. Answer: 19.

$$\frac{A_{n+1}}{A_n} = \frac{1}{n+1}\,\frac{20^{n+1}+11^{n+1}}{20^n+11^n} = \frac{20 + 11\cdot(\frac{11}{20})^n}{(n+1)(1+(\frac{11}{20})^n)}.$$

Then $A_{n+1} < A_n$ if $n > 10 + \dfrac{9}{1+(\frac{11}{20})^n}$; and $A_{n+1} > A_n$ if $n < 10 + \dfrac{9}{1+(\frac{11}{20})^n}$.

Note that $10 + \dfrac{9}{1+(\frac{11}{20})^n} < 10 + 9 = 19$. So $n \ge 19$ implies $A_n > A_{n+1}$.

If $10 \le n \le 18$, then $n \le 10+8 < 10+\dfrac{9}{1+(\frac{11}{20})^n}$; if $n < 10$, then $n < 10+\dfrac{9}{1+(\frac{11}{20})^n}$. Hence, $n \le 18$ implies $A_n < A_{n+1}$.

23. Answer: 169.

Let a_n be the number of ways to pave a block of $1 \times n$. Then $a_n = a_{n-1} + a_{n-2} + a_{n-4}$ with initial conditions $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_4 = 6$. Then

$$a_5 = a_4 + a_3 + a_1 = 10, \quad a_6 = a_5 + a_4 + a_2 = 18,$$
$$a_7 = a_6 + a_5 + a_3 = 31, \quad a_8 = a_7 + a_6 + a_4 = 55,$$
$$a_9 = a_8 + a_7 + a_5 = 96, \quad a_{10} = a_9 + a_8 + a_6 = 169.$$

24. Answer: 288.

Consider the grid below. Suppose the left-top 2×2 sub-grid is filled in with $1, 2, 3, 4$.

If x, y, z, w are all distinct, then there are no other numbers to place in a; if $\{x, y\} = \{z, w\}$, then x', y', z, w are all distinct, and there are no other numbers for a'.

Note that $\{x, x'\} = \{1, 2\}$, $\{y, y'\} = \{3, 4\}$, $\{z, z'\} = \{2, 4\}$ and $\{w, w'\} = \{1, 3\}$. Among these $2^4 = 16$ choices, 4 of them are impossible — $\{x, y\} = \{z, w\} = \{1, 4\}$ or $\{2, 3\}$, $\{x, y\} = \{1, 4\}$ and $\{z, w\} = \{2, 3\}$, $\{x, y\} = \{2, 3\}$ and $\{z, w\} = \{1, 4\}$.

For each of the remaining 12 cases, x', y', z', w' are uniquely determined, so is the right-bottom sub-grid:

$$\{a\} = \{1, 2, 3, 4\} - \{x, y\} \cup \{z, w\},$$
$$\{b\} = \{1, 2, 3, 4\} - \{x, y\} \cup \{z', w'\},$$
$$\{a'\} = \{1, 2, 3, 4\} - \{x', y'\} \cup \{z, w\},$$
$$\{b'\} = \{1, 2, 3, 4\} - \{x', y'\} \cup \{z', w'\}.$$

Recall that that there are $4! = 24$ permutations in the left-top grid. Hence, there are $24 \times 12 = 288$ solutions.

1	2	x	x'
3	4	y	y'
z	w	a	a'
z'	w'	b	b'

25. Answer: 14.

If 13th of January falls on a particular day, represented by 0, then the 13th of February falls 3 days later, represented by $0 + 31 \equiv 3 \pmod 7$.

Case 1: The consecutive two years are non-leap years.

Jan	Feb	Mar	Apr	Mar	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0	3	3	6	1	4	$\boxed{6}$	2	5	0	3	5
1	4	4	0	2	5	0	3	$\boxed{6}$	1	4	6

Case 2: The first year is a leap year.

Jan	Feb	Mar	Apr	Mar	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0	3	4	0	2	5	$\boxed{0}$	3	6	1	4	6
2	5	5	1	3	6	1	4	$\boxed{0}$	2	5	0

Case 3: The second year is a leap year.

Jan	Feb	Mar	Apr	Mar	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0	3	3	6	1	4	$\boxed{6}$	2	5	$\textcircled{0}$	3	5
1	4	5	1	3	$\boxed{6}$	1	4	$\textcircled{0}$	2	5	0

From these tables we see that the answer is 14. The longest time period occurs when the Friday of 13th falls in July of the first year and in September of the second year, while the second year is not a leap year.

26. Answer: 350.

By considering the numbers of apples in the packages, there are 3 cases:

1) $(4,1,1,1)$. $\binom{7}{4} = \frac{7 \times 6 \times 5 \times 4}{4 \times 3 \times 2 \times 1} = 35.$

2) $(3,2,1,1)$. $\binom{7}{3}\binom{4}{2} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = \times \frac{4 \times 3}{2 \times 1} = 35 \times 6 = 210.$

3) $(2,2,2,1)$. $\frac{1}{3!}\binom{7}{2}\binom{5}{2}\binom{3}{2} = \frac{1}{6} \times \frac{7 \times 6}{2 \times 1} \times \frac{5 \times 4}{2 \times 1} \times \frac{3 \times 2}{2 \times 1} = 105.$

So the total number of ways is $35 + 210 + 105 = 350$.

27. Answer: 19.

Note that $8 + 12 = 20$, $5 + 8 + 8 = 21$, $5 + 5 + 12 = 22$, $5 + 5 + 5 + 8 = 23$, $8 + 8 + 8 = 24$. If $n \geq 25$, write $n = 5k + m$ where $20 \leq m \leq 24$ and k is a positive integer. So any amount ≥ 25 can be paid exactly using coupons.

However, 19 cannot be paid exactly using these three types of coupons.

28. Answer: 10301.

The broken line is constructed using "L", with lengths $2, 4, 6, \ldots, 200$. The last "L" is $100 + 101 = 201$. Then the total length is $2(1 + 2 + 3 + \cdots + 100) + 201 = 10301$.

29. Answer: 15.

Let $P(x) = x^{a_1} + \cdots + x^{a_6}$ and $Q(x) = x^{b_1} + \cdots + x^{b_6}$. Then

$$P(x)Q(x) = (x + x^2 + \cdots + x^6)^2 = x^2(1+x)^2(1+x+x^2)^2(1-x+x^2)^2.$$

Note that $P(0) = Q(0) = 0$ and $P(1) = Q(1) = 6$, $x(1+x)(1+x+x^2)$ is a common divisor of $P(x)$ and $Q(x)$.

Since they are not normal dice,

$$P(x) = x(1+x)(1+x+x^2) = x + 2x^2 + 2x^3 + x^4,$$
$$Q(x) = x(1+x)(1+x+x^2)(1-x+x^2) = x + x^3 + x^4 + x^5 + x^6 + x^8.$$

So the numbers of the first dice are $1, 2, 2, 3, 3, 4$ and that of the second dice are $1, 3, 4, 5, 6, 8$.

Then $a_1 + \cdots + a_6 = 1 + 2 + 2 + 3 + 3 + 4 = 15$.

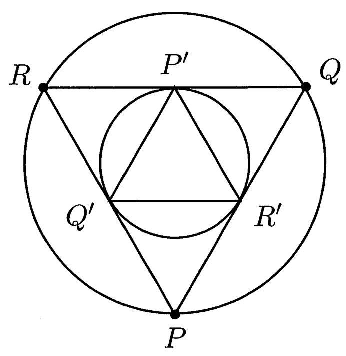
30. Answer: 109.

$2 \cdot PA + 3 \cdot PB + 5 \cdot PC = 2(PA + PC) + 3(PB + PC) \geq 2 \cdot AC + 3 \cdot BC = 2 \cdot 17 + 3 \cdot 25 = 109.$

The equality holds if and only if $P = C$.

31. Answer: 4.

Given an equilateral triangle PQR. Let C_1 be its circumscribed circle and C_2 its inscribed circle. Suppose QR, RP, PQ are tangent to C_2 at P', Q', R', respectively. The area of triangle PQR is 4 times the area of triangle $P'Q'R'$. So the area of C_1 is also 4 times the area of C_2.



32. Answer: 98.

Since $y = x^2$ and $y = -x - 4$ do not intersect, A and B must lie on a line parallel to $y = -x - 4$, namely, $y = -x + c$. The distance from $(0, -4)$ to $y = -x + c$ is

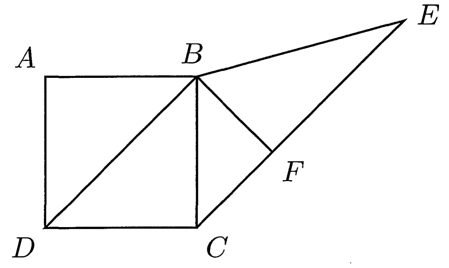
$$8\sqrt{2} = \frac{|(0) + (-4) - c|}{\sqrt{1+1}}.$$

So $c = 12$ or $c = -20$ (ignored). Substitute $y = -x + 12$ into the parabola: $x^2 = -x + 12 \Rightarrow x = 3, -4$. So A is $(3, 9)$ and B is $(-4, 16)$. Then

$$|AB|^2 = (3 - (-4))^2 + (9 - 16)^2 = 98.$$

33. Answer: 30.

Draw $BF \perp CE$, where F is on CE. If $AB = 1$, then $BF = \frac{\sqrt{2}}{2}$ and $BE = \sqrt{2}$. Thus $\angle E = 30°$.



34. Answer: 4022.

Set $DP = GP = a$, $IP = FP = b$, $EP = HP = c$. Then

$$DE + FG + HI = (a + c) + (a + b) + (b + c) = 2(a + b + c) = 2 \times 2011 = 4022.$$

35. Answer: 10.

By given, $BD + 2 + AE = BD + DC$. So $2 + AE = DC$.

Note that $AB^2 + BE^2 = AE^2$ and $BD^2 + BC^2 = DC^2$. Then

$$(2 + BD)^2 + 3^2 = AE^2, \quad BD^2 + 7^2 = (AE + 2)^2.$$

$4(AE + BD) = 32$. Then $AE + BA = \frac{32}{4} + 2 = 10$.