Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2015

Senior Section (Round 1)

Wednesday, 3 June 2015 | **0930 – 1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x. *For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Find the exact value of $\dfrac{\sqrt{50}+7}{\sqrt{50}-7}+\dfrac{\sqrt{50}-7}{\sqrt{50}+7}$.

   (A) 197 (B) 198 (C) 199 (D) 200 (E) 201

2. Simplify $\dfrac{3^x+63}{21^{x-2}+7^{x-1}}$.

   (A) $\dfrac{3}{7^x}$ (B) $\dfrac{9\times 49}{7^x}$ (C) $\dfrac{9}{7^x}$ (D) $\dfrac{9\times 7}{7^x}$ (E) $\dfrac{3\times 7}{7^x}$

3. Suppose m and n are real numbers such that the roots of the equation $2x^2-mx+8=0$ are α and β while the roots of the equation $5x^2-10x+5n=0$ are $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$. Find the value of mn.

   (A) $\dfrac{1}{4}$ (B) 4 (C) 8 (D) 12 (E) 16

4. Find the largest number among the following numbers:

   (A) $\sqrt{8}+\sqrt{8}$ (B) $\sqrt{7}+\sqrt{9}$ (C) $\sqrt{6}+\sqrt{10}$ (D) $\sqrt{5}+\sqrt{11}$ (E) $\sqrt{4}+\sqrt{12}$

5. Which of the following is true?

   (A) $\cos 221^\circ > \sin 319^\circ$ (B) $\cos 139^\circ > \sin 221^\circ$ (C) $\sin 139^\circ > \cos 41^\circ$
   (D) $\sin 221^\circ > \cos 139^\circ$ (E) $\sin 41^\circ > \cos 319^\circ$

6. Find the smallest number among the following numbers:

   (A) $\log_{2015} 2016$ (B) $\log_{2016} 2017$ (C) $\log_{2017} 2018$
   (D) $\log_{2018} 2019$ (E) $\log_{2019} 2020$

7. If x and y are non-zero numbers such that $x>y$, which of the following is always true?

   (A) $\dfrac{1}{x}<\dfrac{1}{y}$ (B) $\dfrac{x}{y}>1$ (C) $|x|>|y|$ (D) $\dfrac{1}{xy^2}>\dfrac{1}{x^2y}$ (E) $\dfrac{x}{y}>\dfrac{y}{x}$

8. If $f(x)=2|x^2-1|-x-1$, where $0\le x\le \dfrac{3}{2}$, find the range of $f(x)$.

   (A) $-3\le f(x)\le 1$ (B) $-3\le f(x)\le \dfrac{3}{2}$ (C) $-2\le f(x)\le 1$
   (D) $-2\le f(x)\le \dfrac{3}{2}$ (E) $-2\le f(x)\le 2$

9. How many prime numbers p satisfy the inequality

$$\left|2 + \log_{3/p}(3p^2)\right| > \frac{3}{2} \text{ ?}$$

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

10. When a polynomial $f(x)$ is divided by $(x-1)$ and $(x+5)$, the remainders are -6 and 6 respectively. Let $r(x)$ be the remainder when $f(x)$ is divided by x^2+4x-5. Find the value of $r(-2)$.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 5

Short Questions

11. Find the value of

$$\frac{2+\log_2 3}{1+\log_2 3} + \frac{3+\log_3 4}{1+\log_3 2}.$$

12. Find the absolute value of the coefficient of $\frac{1}{x}$ in the expansion of

$$\left(2x^2 - \frac{1}{x}\right)^{10}.$$

13. If $a+b = \frac{25}{4}$ and $(1+\sqrt{a})(1+\sqrt{b}) = \frac{15}{2}$, find the value of ab.

14. Find the value of p if there is a unique value of x satisfying the equation $p^{2x+1}+1 = \sqrt{44}e^{x\ln p}$.

15. Suppose x and y are real numbers such that x^2 and y^2 are positive integers. Find the maximum value of $x^2 - xy$ if

$$(3x^2 - y^2)^2 + (x^2+y^2)^2 = 72.$$

16. Find the smallest positive integer x (measured in degrees) such that

$$\tan(x - 160^\circ) = \frac{\cos 50^\circ}{1 - \sin 50^\circ}.$$

17. Find the smallest positive integer k such that

$$\frac{1}{\log_{3^k} 2015!} + \frac{1}{\log_{4^k} 2015!} + \cdots + \frac{1}{\log_{2015^k} 2015!} > 2015.$$

18. Let $a < b < c$ be the three real roots of the equation

$$3x^3 - 35x^2 + 500 = 0.$$

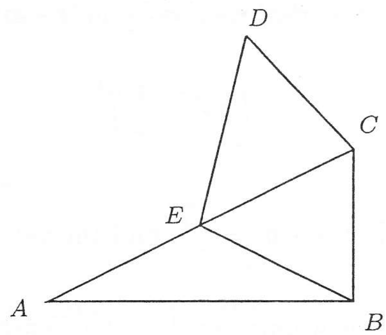
Find $\frac{500^2}{(bc)^2} + \frac{500^2}{(ac)^2} + \frac{500^2}{(ab)^2}$.

19. A function f satisfies $f(x) + f(3x) = x^2 + 1$ for all real numbers x. If $f(2) + f(18) = 6$, determine the value of $f(6)$.

20. It is given that n consecutive positive even integers, the smallest of which is x, have a sum of 154. Find the smallest possible value of x.

21. Find the largest nonnegative integer n such that 2^n is a factor of $\lfloor\sqrt{2}\rfloor \times \lfloor\sqrt{3}\rfloor \times \lfloor\sqrt{4}\rfloor \times \cdots \times \lfloor\sqrt{99}\rfloor$.

22. In the diagram below, $\angle CBA = 90°$, $\angle DCE = 2\angle CAB$ and $AC = 2CD = 2CE =$ 20 meter. Find the maximum possible area (in meter^2) of the quadrilateral $BCDE$.



23. Let $N = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{2015}$. Find the last digit of the number

$$(9 + N)^N.$$

24. Consider an equilateral triangle in which each side has length 1 centimetre. What is the smallest number of points that must be chosen from the region enclosed by the triangle (including the boundary) so that at least two of these points have distance of at most $\frac{1}{2}$ centimetre between them.

25. Find the value of

$$\frac{\cot^3 75° + \tan^3 75°}{\cot 75° + \tan 75°}.$$

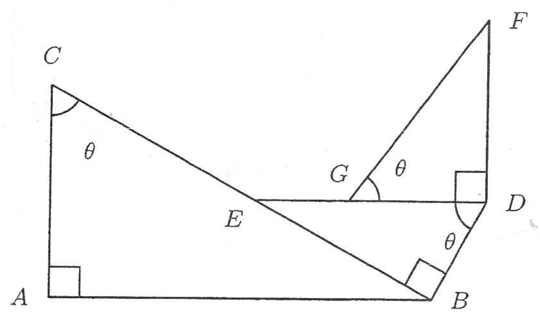
26. Find a 5-digit number so that its digits is completely reversed when multiplying it with some integer n, where $2 \leq n \leq 8$.

27. Consider the following equations:

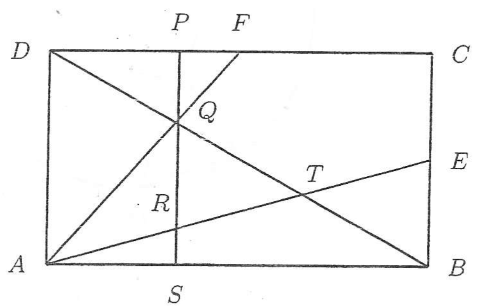
$$\frac{1}{x^2} = \frac{1}{y} + \frac{1}{z},$$
$$\frac{1}{y^2} = \frac{1}{x} + \frac{1}{z},$$
$$\frac{1}{z^2} = \frac{1}{x^2} + \frac{1}{y^2}.$$

Find the sum of $(x+1)^4(y+1)^4$ over all possible ordered triples (x, y, z) that satisfy the above three equations simultaneously.

28. The diagram below shows three right-angled triangles, where $BC = 14$, $GF = 10$, $DE = 7$ and $\angle BCA = \angle BDE = \angle FGD = \theta$. Find the maximum possible value of $AB + BD + DF$.

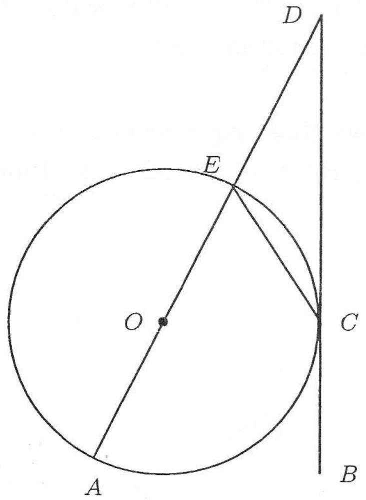


29. The diagram below shows a rectangle $ABCD$ such that E is the midpoint of BC and F is the midpoint of CD. The diagonal BD intersects AF and AE at Q and T respectively. The vertical line PS passing through Q is perpendicular to AB and intersects AE at R. It is also given that $AB = CD = 12$ cm and $BC = AD = 6$ cm. Find the area of the triangle $\triangle QRT$ in cm^2.

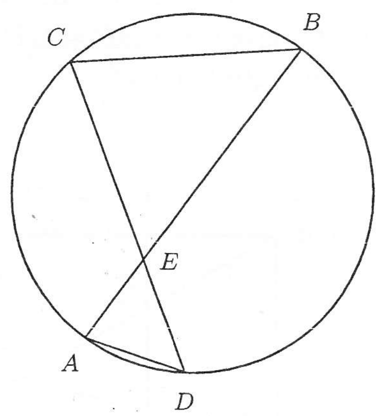


30. Find the minimum value of $13\sec\theta - 9\sin\theta\tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

31. In the figure below, the line BD is tangent to the circle at C. The line AD passes through the centre O of the circle and intersects the circle at E. It is given that $\angle CDE = 34°$ and $\angle DCE = x°$. Find the value of x.

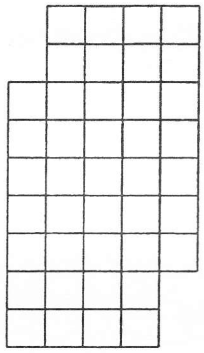


32. In the figure below, A, B, C and D are points on the circle such that the straight lines AB and CD intersect at E. Let $[BCE]$ and $[ADE]$ denote the areas of the triangles $\triangle BCE$ and $\triangle ADE$ respectively. If $\frac{[BCE]}{[ADE]} = 25$ and $AE = 1$ cm, find the length of the line CE in cm.



33. In how many ways can a group of 8 different guests (consisting of 4 males and 4 females) be seated at a round table with 8 seats such that there are exactly 3 males who are seated next to each other?

34. Find the number of rectangles that can be formed from the gridlines of the board as shown in the figure below.



35. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$, and let $\mathcal{A} = \{F_1, F_2, \ldots, F_n\}$ be a collection of distinct subsets of X such that the intersection $F_i \cap F_j$ contains exactly one element whenever $i \neq j$. For each $i \in X$, let r_i be the number of elements in $\mathcal{A}$ which contains i. Suppose $r_1 = r_2 = 1$, $r_3 = r_4 = r_5 = r_6 = 2$ and $r_7 = 4$. Find the value of $n^2 - n$.