Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2015

Junior Section (Round 1)

Wednesday, 3 June 2015 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x. *For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.
9. *Throughout this paper, let* $\overline{a_{n-1}a_{n-2}\ldots a_0}$ *denote an* n*-digit number with the digits* a_i *in the corresponding position, i.e.* $\overline{a_{n-1}a_{n-2}\ldots a_0} = a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + \cdots + a_0 10^0$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Among the five numbers $\frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{6}{11}$ and $\frac{13}{21}$, which one has the smallest value?

(A) $\frac{5}{9}$ (B) $\frac{4}{7}$ (C) $\frac{3}{5}$ (D) $\frac{6}{11}$ (E) $\frac{13}{21}$

2. Adrian, Billy, Christopher, David and Eric are the five starters of a school's basketball team. Two among the five shoot with their left hand while the rest shoot with their right hand. Among the five, only two are more than 1.8 metres in height. Adrian and Billy shoot with the same hand, but Christopher and David shoot with different hands. Billy and Christopher are respectively the shortest and tallest member of the team, while Adrian and David have the same height. Who is more than 1.8 metres tall and shoots with his left hand?

(A) None (B) Only Christopher (C) Only Eric
(D) Christopher and Eric (E) Not enough information to ascertain

3. How many ways are there to arrange 3 identical blue balls and 2 identical red balls in a row if the two red balls must always be next to each other?

(A) 2 (B) 4 (C) 5 (D) 10 (E) 20

4. If a, b and c are positive real numbers such that

$$\frac{a}{a+b} = \frac{a+b}{a+b+c} = \frac{c}{b+c},$$

then $\frac{a}{b}$ equals

(A) $\frac{\sqrt{3}-1}{2}$ (B) 1 (C) $\sqrt{2}$ (D) $\frac{1+\sqrt{5}}{2}$ (E) None of the above

5. In the figure below, each distinct letter represents a unique digit such that the arithmetic sum holds. What is the digit represented by the letter B?

$$\begin{array}{rccccc} & & M & A & T & H \\ + & & M & A & T & H \\ \hline & H & A & B & I & T. \end{array}$$

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

6. Find the minimum value of the function $2015 - \frac{10}{x^2 - 4x + 5}$.

(A) 2000 (B) 2005 (C) 2010 (D) 2013 (E) None of the above

7. It is known that 99900009 is the product of four consecutive odd numbers. Find the sum of squares of these four odd numbers.

(A) 40000 (B) 40010 (C) 40020 (D) 40030 (E) 40040

8. The lengths of the sides of a triangle are x^2, $22-x$ and $x-2$. The total number of possible integer values of x is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

9. Find the value of $\left(4\sqrt{4+2\sqrt{3}}-\sqrt{49+8\sqrt{3}}\right)^2$.

(A) $3\sqrt{3}$ (B) 6 (C) $4\sqrt{3}$ (D) 9 (E) $4\sqrt{3}+3$

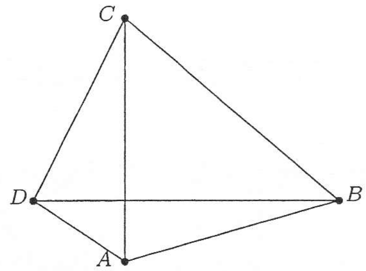
10. If x and y satisfy the equation $2x^2+3y^2=4x$, the maximum value of $10x+6y^2$ is

(A) 2 (B) $\frac{9}{2}$ (C) 20 (D) $\frac{81}{4}$ (E) None of the above

Short Questions

11. In a shop, the price of a particular type of toy is a whole number greater than \$100. The total sales of this type of toy on a particular Saturday and Sunday were \$1518 and \$2346 respectively. Find the total number of toys sold on these two days.

12. A quadrilateral $ABCD$ has perpendicular diagonals AC and BD with lengths 8 and 10 respectively. Find the area of the quadrilateral.



13. Find the smallest positive integer that is divisible by every integer from 1 to 12.

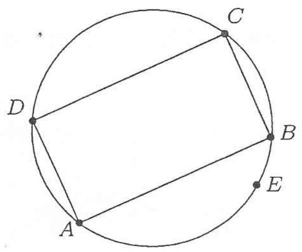
14. Results of a school wide vote for the president of the student council showed that two-fifths of the vote went to Alice, five-twelfths of the vote went to Bobby and the remaining 33 votes went to Charlie. If every student voted, how many students were there in the school?

15. Evaluate the sum $\left\lfloor \frac{2^2}{3} \right\rfloor + \left\lfloor \frac{3^2}{4} \right\rfloor + \left\lfloor \frac{4^2}{5} \right\rfloor + \cdots + \left\lfloor \frac{99^2}{100} \right\rfloor$.

16. A quiz was given to a class in which one quarter of the students are male. The class average score on the quiz was 16.5. Excluding three male students whose total score was 21, the average score of all the other students would be 17 while the average score of all the other male students would be 13.25. Find the average score of female students.

17. If $N = 1001^4$, find the sum of all the digits of N.

18. In the diagram below, a rectangle $ABCD$ is inscribed in a circle and E is a point on the circumference of the circle. Given that $|AE|^2 + |BE|^2 + |CE|^2 + |DE|^2 = 450$ and $|AB| \times |BC| = 108$, find the perimeter of the rectangle.



19. Find the largest prime factor of 999936.

20. Find the value of $p + q$, where p and q are two positive integers such that p and q have no common factor larger than 1 and

$$\frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90} = \frac{p}{q}.$$

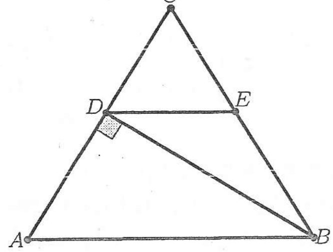
21. Find the value of $\sqrt{(98 \times 100 + 2)(100 \times 102 + 2) + (100 \times 2)^2}$.

22. Find the value of $x^3 - x^2 - 3x + 2015$ if $x = \sqrt{2} + 1$.

23. In the diagram below, ABC is an isosceles triangle with $|AC| = |BC|$. The point D lies on AC such that BD is perpendicular to AC. The point E lies on BC such that DE is parallel to AB. If $|AD| = 3$, $|AB| = 5$ and

$$\frac{\text{Area } \triangle CDE}{\text{Area } \triangle CAB} = \frac{m}{n},$$

where m and n are positive integers with no common factor larger than 1, find the value of $m + n$.



24. Find the remainder when 2015^{2015} is divided by 7.

25. Find the total number of integers in the sequence $20, 21, 22, 23, \cdots, 2014, 2015$ which are multiples of 3 but not multiples of 5.

26. In the quadrilateral $ABCD$, $|AB| = 8$, $|BC| = 1$, $\angle DAB = 30^\circ$ and $\angle ABC = 60^\circ$. If the area of the quadrilateral is $5\sqrt{3}$, find the value of $|AD|^2$.

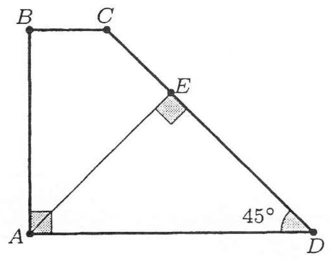
27. Find the total number of six-digit integers of the form $\overline{x2015y}$ which are divisible by 33.

28. The line whose equation is $2x+y = 100$ meets the y-axis at A. B is the point on the x-axis such that AB is perpendicular to the line and C lies on AB such that OC is perpendicular to AB, where O is the origin. D is the foot of perpendicular from C to the x-axis. Find the area of triangle OCD.

29. If $xy < 0$, $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 40$ and $x + y = \dfrac{1}{3}$, find the value of $\dfrac{1}{x^4} + \dfrac{1}{y^4}$.

30. Let n be a positive integer. Assume that the sum of n and 7 is a multiple of 8 but the difference of n and 7 is a multiple of 14. Find the largest possible value of n such that $n < 10000$.

31. In the diagram below, $ABCD$ is a trapezium with $\angle D = 45^\circ$, $\angle A = 90^\circ$, $|BC| = 1$ and $|CD| = 2\sqrt{2}$. E is a point on CD such that AE is perpendicular to CD, find the value of $4|AE|^2$.



32. A donut shop sells its donuts *only* in packs of 6 (half-dozen) or in packs of 13 (a baker's dozen). So it is impossible to purchase exactly 14 donuts from this shop, since 14 cannot be written as an integral combination of 6 and 13. Find the largest number of donuts that cannot be purchased from this shop.

33. If n is a positive integer such that $n^2 - 7n + 17$ is equal to the product of two consecutive odd integers, find the sum of these two consecutive odd integers.

34. Evaluate the sum

$$\left\lfloor\frac{1}{1}\right\rfloor + \left\lfloor\frac{2}{1}\right\rfloor + \left\lfloor\frac{1}{2}\right\rfloor + \left\lfloor\frac{2}{2}\right\rfloor + \left\lfloor\frac{3}{2}\right\rfloor + \left\lfloor\frac{4}{2}\right\rfloor + \left\lfloor\frac{1}{3}\right\rfloor + \left\lfloor\frac{2}{3}\right\rfloor + \left\lfloor\frac{3}{3}\right\rfloor + \left\lfloor\frac{4}{3}\right\rfloor + \left\lfloor\frac{5}{3}\right\rfloor + \left\lfloor\frac{6}{3}\right\rfloor + \cdots$$

up to the 2015^{th} term.

35. In the diagram below, the area of the rectangle $ABCD$ is 80. The point E lies on the side AB. $DEFG$ is a trapezium with parallel sides EF and DG. C is the midpoint of the side FG. Find the area of the trapezium $DEFG$ if the area of $\triangle AED$ is 23 and the area of the shaded triangle is 5.