Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 2)

Saturday, 4 July 2009 **0900-1330**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let O be the center of the circle inscribed in a rhombus $ABCD$. Points E, F, G, H are chosen on sides AB, BC, CD and DA respectively so that EF and GH are tangent to the inscribed circle. Show that EH and FG are parallel.

2. A palindromic number is a number which is unchanged when the order of its digits is reversed. Prove that the arithmetic progression 18, 37, ... contains infinitely many palindromic numbers.

3. For k a positive integer, define A_n for $n = 1, 2, \ldots,$ by

$$A_{n+1} = \frac{nA_n + 2(n+1)^{2k}}{n+2}, \quad A_1 = 1.$$

Prove that A_n is an integer for all $n \geq 1$, and A_n is odd if and only if $n \equiv 1$ or $2 \pmod 4$.

4. Find the largest constant C such that

$$\sum_{i=1}^{4} (x_i + \frac{1}{x_i})^3 \geq C$$

for all positive real numbers $x_1, \cdots, x_4$ such that

$$x_1^3 + x_3^3 + 3x_1x_3 = x_2 + x_4 = 1.$$

5. Find all integers x, y and z with $2 \leq x \leq y \leq z$ such that

$$xy \equiv 1 \pmod z, \quad xz \equiv 1 \pmod y, \quad yz \equiv 1 \pmod x.$$