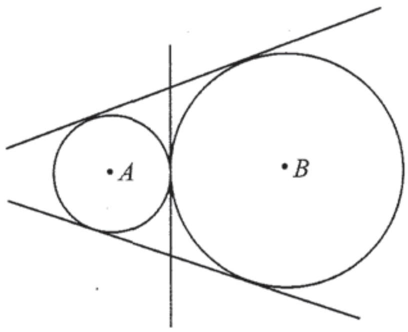
Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section Solutions)

1. Answer: (C)
In the plane π, draw a circle of radius 7 cm centred at A and a circle of radius 26 cm centred at B.



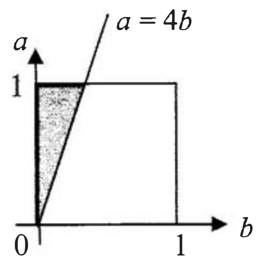
If ℓ is a line on the plane π, and the distance between ℓ and A is 7 cm and the distance between ℓ and B is 26 cm, then ℓ must be tangential to both circles. Clearly, there are 3 lines in the plane that are tangential to both circles, as shown in the figure above.

2. Answer: (A)
We have

$$\begin{aligned} y &= (17^2 - x^2)(19^2 - x^2) \\ &= x^4 - (17^2 + 19^2)x^2 + 17^2 \cdot 19^2 \\ &= x^4 - 650x^2 + 323^2 \\ &= (x^2 - 325)^2 + 323^2 - 325^2 \end{aligned}$$

Hence the smallest possible value of y is $323^2 - 325^2 = (-2)(648) = -1296$.

3. Answer: (A)
The discriminant of the equation is $a - 4b$.
Thus the equation has real roots if and only if $a \geq 4b$.
The shaded part in the figure on the right are all the points with coordinates (a, b) such that $0 < a, b < 1$ and $a \geq 4b$. As the area of the shaded part is $\frac{1}{8}$, it follows that the required probability is $\frac{1}{8}$.



4. Answer: (D)

If $x \le 0$, then $|x| = -x$, and we obtain from $|x| + x + 5y = 2$ that $y = \frac{2}{5}$. Thus y is positive, so $|y| - y + x = 7$ gives $x = 7$, which is a contradiction since $x \le 0$. Therefore we must have $x > 0$. Consequently, $|x| + x + 5y = 2$ gives the equation

$$2x + 5y = 2. \qquad (1)$$

If $y \ge 0$, then $|y| - y + x = 7$ gives $x = 7$. Substituting $x = 7$ into $|x| + x + 5y = 2$, we get $y = -\frac{12}{5}$, which contradicts $y \ge 0$. Hence we must have $y < 0$, and it follows from the equation $|y| - y + x = 7$ that

$$x - 2y = 7. \qquad (2)$$

Solving equations (1) and (2) gives $x = \frac{13}{3}$, $y = -\frac{4}{3}$. Therefore $x + y = 3$.

5. Answer: (C)

$\sin A = \frac{3}{5}$ implies that $\cos A = \frac{4}{5}$ or $-\frac{4}{5}$, and $\cos B = \frac{5}{13}$ implies that $\sin B = \frac{12}{13}$, since $0 < B < 180^\circ$.

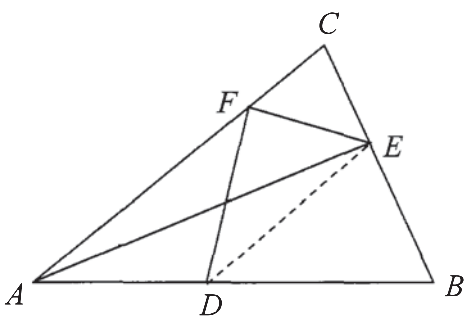
If $\cos A = -\frac{4}{5}$, then $\sin(A+B) = \sin A \cos B + \cos A \sin B = \frac{3}{5}\cdot\frac{5}{13} - \frac{4}{5}\cdot\frac{12}{13} < 0$, which is not possible since $0 < A + B < 180^\circ$ in a triangle. Thus we must have $\cos A = \frac{4}{5}$. Consequently, since $C = 180^\circ - (A + B)$, we have

$$\cos C = -\cos(A+B) = -\cos A \cos B + \sin A \sin B$$
$$= -\frac{4}{5}\cdot\frac{5}{13} + \frac{3}{5}\cdot\frac{12}{13} = \frac{16}{65}.$$

6. Answer: (E)

Since area of triangle ABE is equal to area of quadrilateral $DBEF$, we see that area of ΔDEA = area of ΔDEF. This implies that DE is parallel to AF.

Thus $\frac{CE}{CB} = \frac{AD}{AB} = \frac{3}{8}$. Since $\frac{\text{area of } \Delta AEC}{\text{area of } \Delta ABC} = \frac{CE}{CB}$, it follows that area of $\Delta AEC = \frac{3}{8} \times 40 = 15\ \text{cm}^2$.



7. Answer: (C)

First note that $(n-2)!+(n-1)!+n!=(n-2)![1+(n-1)+n(n-1)]=n^2(n-2)!$.
Therefore the given series can be written as

$$\sum_{n=3}^{22}\frac{n}{n^2(n-2)!}=\sum_{n=3}^{22}\frac{1}{n(n-2)!}=\sum_{n=3}^{22}\frac{n-1}{n(n-1)(n-2)!}$$

$$=\sum_{n=3}^{22}\frac{n-1}{n!}=\sum_{n=3}^{22}\left(\frac{1}{(n-1)!}-\frac{1}{n!}\right).$$

Summing the telescoping series, we obtain $\frac{1}{2!}-\frac{1}{22!}$.

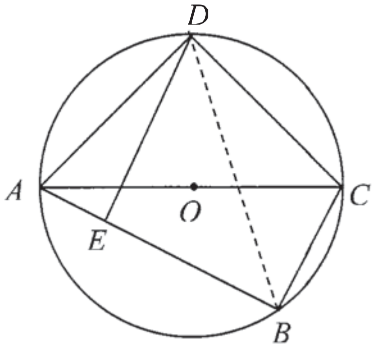
8. Answer: (D)

There are $\binom{8}{4}=70$ ways of putting 4 identical red buttons and 4 identical blue buttons in the envelopes. Since $1+2+3+\cdots+8=36$, there are 8 cases where the sum of the numbers on the envelopes containing the red buttons is equal to 18 (which is also equal to the sum of the numbers on the envelopes containing the blue buttons), namely, (8, 7, 2, 1), (8, 6, 3, 1), (8, 5, 4, 1), (8, 5, 3, 2), (7, 6, 4, 1), (7, 6, 3, 2), (7, 5, 4, 2) and (6, 5, 4, 3). Hence it follows that the required number of ways is $\frac{70-8}{2}=31$.

9. Answer: (B)

Let the angles of the acute-angled triangle be x°, y°, $3x^\circ$, where the smallest angle is x°. Then we have $x+y+3x=180$ and $0<x\le y\le 3x<90$. From the inequalities $x\le y\le 3x$, we obtain $5x\le x+y+3x\le 7x$, and hence it follows from the first equation that $\frac{180}{7}\le x\le 36$. Since x is an integer and $3x<90$, we deduce that $x=26, 27, 28, 29$. Hence there are 4 acute-angled triangles whose angles are respectively $(26^\circ, 76^\circ, 78^\circ)$, $(27^\circ, 72^\circ, 81^\circ)$, $(28^\circ, 68^\circ, 84^\circ)$ and $(29^\circ, 64^\circ, 87^\circ)$.

10. Answer: (B)



Let r be the radius of the circle with centre O.
Since $AD = DC$ and $\angle ADC = 90^\circ$, $\angle ACD = 45^\circ$. Thus $\angle ABD = 45^\circ$. As $\angle DEB = 90^\circ$, this implies that $DE = BE$. Let $x = DE = BE$. Since $BC /\!/ ED$ and

area of quadrilateral $ABCD$ = area of ΔAED + area of ΔEBD + area of ΔBCD,

we have

$$24 = \frac{1}{2}\cdot(AB - x)x + \frac{1}{2}x^2 + \frac{1}{2}\cdot BC\cdot x = \frac{1}{2}(AB + BC)x. \quad (1)$$

On the other hand, as OD is perpendicular to AC, and

area of quadrilateral $ABCD$ = area of ΔABC + area of ΔACD,

we have

$$24 = \frac{1}{2}\cdot AB\cdot BC + \frac{1}{2}AC\cdot OD = \frac{1}{2}\cdot AB\cdot BC + r^2. \quad (2)$$

Now equation (2) and $AB^2 + BC^2 = AC^2 = 4r^2$ imply that

$$(AB + BC)^2 = 4r^2 + 2\cdot AB\cdot BC = 4r^2 + 4(24 - r^2) = 96.$$

Therefore $AB + BC = 4\sqrt{6}$. Hence from equation (1), we obtain $x = 2\sqrt{6}$.

11. Answer: 91

Let $a = 2008$. Then

$$\begin{aligned}(2008^3 + (3 \times 2008 \times 2009) + 1)^2 &= (a^3 + 3a(a+1) + 1)^2 \\ &= (a^3 + 3a^2 + 3a + 1)^2 \\ &= (a+1)^6 = 2009^6 = 7^{12}\cdot 41^6.\end{aligned}$$

Hence the number of positive divisors is $(12 + 1)(6 + 1) = 91$.

12. Answer: 3

$$\frac{1}{1+\log_{a^2b}\left(\frac{c}{a}\right)}+\frac{1}{1+\log_{b^2c}\left(\frac{a}{b}\right)}+\frac{1}{1+\log_{c^2a}\left(\frac{b}{c}\right)}$$
$$=\frac{1}{\log_{a^2b}(a^2b)+\log_{a^2b}\left(\frac{c}{a}\right)}+\frac{1}{\log_{b^2c}(b^2c)+\log_{b^2c}\left(\frac{a}{b}\right)}+\frac{1}{\log_{c^2a}(c^2a)+\log_{c^2a}\left(\frac{b}{c}\right)}$$
$$=\frac{1}{\log_{a^2b}(abc)}+\frac{1}{\log_{b^2c}(abc)}+\frac{1}{\log_{c^2a}(abc)}$$
$$=\log_{abc}(a^2b)+\log_{abc}(b^2c)+\log_{abc}(c^2a)$$
$$=\log_{abc}(abc)^3=3.$$

13. Answer: 2008

Observe that $n!\times n = n!\times(n+1-1)=(n+1)!-n!$. Therefore

$$(1!\times 1)+(2!\times 2)+(3!\times 3)+\cdots+(286!\times 286)$$
$$=(2!-1!)+(3!-2!)+(4!-3!)+\cdots+(287!-286!)$$
$$=287!-1.$$

Since $2009 = 287\times 7$, $287!-1\equiv -1\equiv 2008 \pmod{2009}$. It follows that the remainder is 2008.

14. Answer: 5

Let $x=\left(25+10\sqrt{5}\right)^{1/3}+\left(25-10\sqrt{5}\right)^{1/3}$. Then

$$x^3=\left(25+10\sqrt{5}+25-10\sqrt{5}\right)+3\left(25^2-100(5)\right)^{1/3}\left[\left(25+10\sqrt{5}\right)^{1/3}+\left(25-10\sqrt{5}\right)^{1/3}\right],$$

which gives $x^3=50+15x$, or $(x-5)(x^2+5x+10)=0$. This equation admits only one real root $x = 5$.

15. Answer: 1024

$a=\dfrac{1+\sqrt{2009}}{2}$ gives $(2a-1)^2=2009$, which simplified to $a^2-a=502$. Now

$$\left(a^3-503a-500\right)^{10}=\left(a(a^2-a-502)+a^2-a-500\right)^{10}$$
$$=\left(a(a^2-a-502)+(a^2-a-502)+2\right)^{10}$$
$$=(0+0+2)^{10}=1024.$$

16. Answer: 1

Since DE is the angle bisector of $\angle ADB$, we have $\frac{AE}{EB}=\frac{AD}{BD}$. Similarly, since DF is the angle bisector of $\angle ADC$, $\frac{AF}{CF}=\frac{AD}{DC}$. Hence $\frac{AE}{EB}\cdot\frac{BD}{DC}\cdot\frac{CF}{FA}=1$.

17. Answer: 8

First note that if $A+B=45^\circ$, then $1=\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$, and so $1-\tan A-\tan B=\tan A\tan B$. Consequently,

$$(\cot A-1)(\cot B-1)=\frac{1-\tan A-\tan B+\tan A\tan B}{\tan A\tan B}=\frac{2\tan A\tan B}{\tan A\tan B}=2.$$

Hence

$$\begin{aligned}&(\cot 25^\circ-1)(\cot 24^\circ-1)(\cot 23^\circ-1)(\cot 22^\circ-1)(\cot 21^\circ-1)(\cot 20^\circ-1)\\&=(\cot 25^\circ-1)(\cot 20^\circ-1)(\cot 24^\circ-1)(\cot 21^\circ-1)(\cot 23^\circ-1)(\cot 22^\circ-1)\\&=8.\end{aligned}$$

18. Answer: 602

Note that $ab+a+b=(a+1)(b+1)-1$. Thus $ab+a+b$ is a multiple of 7 if and only if $(a+1)(b+1)\equiv 1 \pmod 7$.

Let $A=\{1, 2, 3, \ldots, 99, 100\}$, and let $A_i=\{x\in A : x\equiv i \pmod 7\}$ for $i=0, 1, 2, \ldots, 6$. It is easy to verify that for any $x\in A_i$ and $y\in A_j$, where $0\le i\le j\le 6$, $xy\equiv 1 \pmod 7$ if and only if $i=j\in\{1, 6\}$, or $i=2$ and $j=4$, or $i=3$ and $j=5$. Thus we consider three cases.

Case 1: $a+1,\ b+1\in A_i$ for $i\in\{1, 6\}$.

Then $a, b\in A_i$ for $i\in\{0, 5\}$. As $|A_0|=|A_5|=14$, the number of such subsets $\{a, b\}$ is $2\binom{14}{2}=182$.

Case 2: $a+1$ and $b+1$ are contained in A_2 and A_4 respectively, but not in the same set.

Then a and b are contained in A_1 and A_3 respectively, but not in the same set. Since $|A_1|=15$ and $|A_3|=14$, the number of such subsets $\{a, b\}$ is $14\times 15=210$.

Case 3: $a+1$ and $b+1$ are contained in A_3 and A_5 respectively, but not in the same set.

Then a and b are contained in A_2 and A_4 respectively, but not in the same set. Note that $|A_2|=15$ and $|A_4|=14$. Thus the number of such subsets $\{a, b\}$ is $14\times 15=210$.

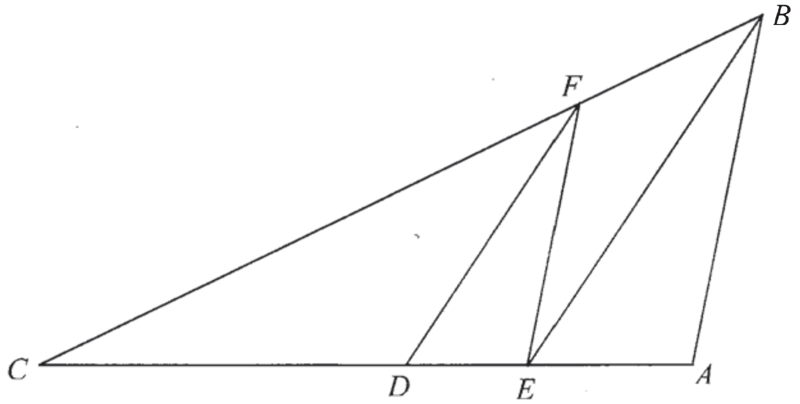
Hence the answer is $182+2\times 210=602$.

19. Answer: 49727

Since $x^2 - 15x + 1 = 0$, $x + \frac{1}{x} = 15$. Therefore

$$x^4 + \frac{1}{x^4} = \left(x + \frac{1}{x}\right)^4 - 4\left(x^2 + \frac{1}{x^2}\right) - 6$$
$$= \left(x + \frac{1}{x}\right)^4 - 4\left(x + \frac{1}{x}\right)^2 + 8 - 6$$
$$= 15^4 - 4 \times 15^2 + 2 = 49727.$$

20. Answer: 24



Since BE bisects $\angle ABC$, we have $AE : EC = AB : BC = 1 : 4$. Furthermore, since $EF /\!/ AB$ and $DF /\!/ EB$, we see that DF bisects $\angle EFC$. Hence $DE : DC = 1 : 4$. Let $AE = x$ and $DE = y$. Then we have $x + y = 13.5$ and $4x = 5y$. Solving the equations yields $x = 7.5$ and $y = 6$. It follows that $CD = 4y = 24$.

21. Answer: 89440

The number of such ordered triples (x, y, z) with $x = y$ is

$$\binom{65}{2} = 2080.$$

The number of such ordered triples (x, y, z) with $x \neq y$ is

$$2 \times \binom{65}{3} = 87360.$$

Hence the answer is $2080 + 87360 = 89440$.

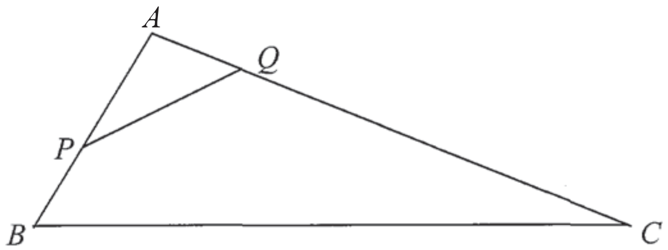
22. Answer: 19901

First note that $\dfrac{1}{a_{n+1}a_n} - \dfrac{1}{a_n a_{n-1}} = \dfrac{1+na_{n-1}a_n}{a_{n-1}a_n} - \dfrac{1}{a_n a_{n-1}} = n$. Therefore

$$\sum_{n=1}^{199}\left(\frac{1}{a_{n+1}a_n} - \frac{1}{a_n a_{n-1}}\right) = \sum_{n=1}^{199} n = \frac{199\times 200}{2} = 19900.$$

Hence $\dfrac{1}{a_{200}a_{199}} = 1 + 19900 = 19901$.

23. Answer: 6



We have $\cos A = \dfrac{AB^2 + AC^2 - BC^2}{2(AB)(AC)} = \dfrac{5^2 + 10^2 - 13^2}{2(5)(10)} = -\dfrac{11}{25}$.

Let $AP = x$ cm and $AQ = y$ cm. Since area of $\Delta APQ = \dfrac{1}{2}xy\sin A$ and area of $\Delta ABC = \dfrac{1}{2}(AB)(AC)\sin A = \dfrac{1}{2}(5)(10)\sin A$, we obtain $\dfrac{xy}{50} = \dfrac{1}{4}$, that is, $xy = \dfrac{25}{2}$.

Hence

$$PQ^2 = x^2 + y^2 - 2xy\cos A = x^2 + \left(\frac{25}{2x}\right)^2 - 25\left(-\frac{11}{25}\right)$$

$$= x^2 + \frac{625}{4x^2} + 11 \geq 2\sqrt{x^2 \cdot \frac{625}{4x^2}} + 11 = 25 + 11 = 36.$$

Consequently, $PQ \geq 6$, with the equality attained when $x = y = \dfrac{5}{\sqrt{2}}$.

24. Answer: 5

Since $x + y = 9 - z$, $xy = 24 - z(x+y) = 24 - z(9-z) = z^2 - 9z + 24$. Now note that x and y are roots of the quadratic equation $t^2 + (z-9)t + (z^2 - 9z + 24) = 0$. As x and y are real, we have $(z-9)^2 - 4(z^2 - 9z + 24) \geq 0$, which simplified to $z^2 - 6z + 5 \leq 0$. Solving the inequality yields $1 \leq z \leq 5$. When $x = y = 2$, $z = 5$. Hence the largest possible value of z is 5.

25. Answer: 357

First put the six 1's in one sequence. Then there are 7 gaps before the first 1, between two adjacent 1's and after the last 1. For each such gap, we can put a single 0 or double 0's (that is, 00).

If there are exactly i double 0's, then there are exactly $6 - 2i$ single 0's, where $i = 0, 1, 2, 3$. Therefore the number of such binary sequences with exactly i double 0's is $\binom{7}{i}\binom{7-i}{6-2i}$. Hence the answers is $\sum_{i=0}^{3}\binom{7}{i}\binom{7-i}{6-2i} = 357$.

26. Answer: 95

$$\frac{\cos 100^\circ}{1-4\sin 25^\circ \cos 25^\circ \cos 50^\circ} = \frac{\cos 100^\circ}{1-2\sin 50^\circ \cos 50^\circ} = \frac{\cos^2 50^\circ - \sin^2 50^\circ}{(\cos 50^\circ - \sin 50^\circ)^2}$$

$$= \frac{\cos 50^\circ + \sin 50^\circ}{\cos 50^\circ - \sin 50^\circ} = \frac{1+\tan 50^\circ}{1-\tan 50^\circ}$$

$$= \frac{\tan 45^\circ + \tan 50^\circ}{1-\tan 45^\circ \tan 50^\circ} = \tan 95^\circ.$$

Hence $x = 95$.

27. Answer: 223

$$\log_{\frac{x}{9}}\left(\frac{x^2}{3}\right) < 6 + \log_3\left(\frac{9}{x}\right)$$

$$\Leftrightarrow \frac{\log_3\left(\frac{x^2}{3}\right)}{\log_3\left(\frac{x}{9}\right)} < 6 + \log_3 9 - \log_3 x$$

$$\Leftrightarrow \frac{\log_3 x^2 - \log_3 3}{\log_3 x - \log_3 9} < 6 + \log_3 9 - \log_3 x.$$

Let $u = \log_3 x$. Then the inequality becomes $\frac{2u-1}{u-2} < 8 - u$, which is equivalent to $\frac{u^2 - 8u + 15}{u-2} < 0$. Solving the inequality gives $u < 2$ or $3 < u < 5$, that is, $\log_3 x < 2$ or $3 < \log_3 x < 5$. It follows that $0 < x < 9$ or $27 < x < 243$. Hence there are 223 such integers.

28. Answer: 79

First observe that

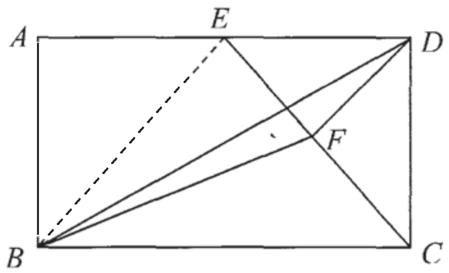
$$\frac{1}{x\sqrt{x+2}+(x+2)\sqrt{x}}=\frac{1}{\sqrt{x}\cdot\sqrt{x+2}}\left(\frac{1}{\sqrt{x}+\sqrt{x+2}}\right)$$
$$=\frac{1}{\sqrt{x}\cdot\sqrt{x+2}}\cdot\frac{\sqrt{x+2}-\sqrt{x}}{(x+2)-x}$$
$$=\frac{1}{2}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+2}}\right).$$

Therefore

$$\frac{1}{9\sqrt{11}+11\sqrt{9}}+\frac{1}{11\sqrt{13}+13\sqrt{11}}+\cdots+\frac{1}{n\sqrt{n+2}+(n+2)\sqrt{n}}$$
$$=\frac{1}{2}\left(\frac{1}{\sqrt{9}}-\frac{1}{\sqrt{11}}\right)+\frac{1}{2}\left(\frac{1}{\sqrt{11}}-\frac{1}{\sqrt{13}}\right)+\cdots+\frac{1}{2}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+2}}\right)=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{\sqrt{n+2}}\right).$$

Now $\frac{1}{2}\left(\frac{1}{3}-\frac{1}{\sqrt{n+2}}\right)=\frac{1}{9}$ yields $n=79$.

29. Answer: 96



Let S be the area of rectangle $ABCD$. Then we have

$$\text{area of } \Delta CDF=\frac{1}{2}\times\text{area of } \Delta CDE=\frac{1}{2}\times\frac{1}{4}S=\frac{1}{8}S.$$

Next we have $\text{area of } \Delta BCF=\frac{1}{2}\times\text{area of } \Delta BCE=\frac{1}{2}\times\frac{1}{2}S=\frac{1}{4}S.$

Now

$$12=\text{area of } \Delta BDF=\text{area of } \Delta BCD-\text{area of } \Delta BCF-\text{area of } \Delta CDF$$
$$=\frac{1}{2}S-\frac{1}{4}S-\frac{1}{8}S=\frac{1}{8}S.$$

Hence the area of rectangle $ABCD = 96\ \text{cm}^2$.

30. Answer: 11754

First note that if every digit in the 6-digit number appears at least twice, then there cannot be four distinct digits in the number. In other words, the number can only be formed by using one digit, two distinct digits or three distinct digits respectively. Therefore we consider three cases.

Case 1: The 6-digit number is formed by only one digit.

Then the number of such 6-digit numbers is clearly 9.

Case 2: The 6-digit number is formed by two distinct digits.

First, the number of such 6-digit numbers formed by two given digits i and j, where $1 \le i < j \le 9$, is

$$\binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 50.$$

Next, the number of such 6-digit numbers formed by 0 and a given digit i, where $1 \le i \le 9$, is

$$\binom{5}{2} + \binom{5}{3} + \binom{5}{4} = 25.$$

Therefore the total number of such 6-digit numbers formed by two distinct digits is

$$\binom{9}{2} \times 50 + 9 \times 25 = 2025.$$

Case 3: The 6-digit number is formed by three distinct digits.

First, the number of such 6-digit numbers formed by three given digits i, j and k, where $1 \le i < j < k \le 9$, is

$$\binom{6}{2} \cdot \binom{4}{2} = 90.$$

Next, the number of such 6-digit numbers formed by 0 and two given digits i and j, where $1 \le i < j \le 9$, is

$$\binom{5}{2} \cdot \binom{4}{2} = 60.$$

Therefore the total number of such 6-digit numbers formed by three distinct digits is

$$\binom{9}{3} \times 90 + \binom{9}{2} \times 60 = 9720.$$

Hence the answer is $9 + 2025 + 9720 = 11754$.

31. Answer: 234

Since $27x + 35y \leq 945$, we have $y \leq \dfrac{945-27x}{35}$. It follows that

$$xy \leq \frac{945x-27x^2}{35} = \frac{27}{35}(35x - x^2) = \frac{27}{35}\left(\left(\frac{35}{2}\right)^2 - \left(x - \frac{35}{2}\right)^2\right).$$

Therefore, if $\left|x - \dfrac{35}{2}\right| \geq \dfrac{5}{2}$, that is, if $x \geq 20$ or $x \leq 15$, then

$$xy \leq \frac{27}{35}\left(\left(\frac{35}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right) < 231.4.$$

If $x = 16$, then $y \leq \dfrac{945-27(16)}{35} \leq 14.7$. Thus $y \leq 14$, and $xy \leq 224$.

Similarly, if $x = 17$, then $y \leq 13$, and $xy \leq 221$.

If $x = 18$, then $y \leq 13$, and $xy \leq 234$.

If $x = 19$, then $y \leq 12$, and $xy \leq 228$.

In conclusion, the maximum value of xy is 234, which is attained at $x = 18$ and $y = 13$.

32. Answer: 65520

Note that

$$(1+x^5+x^7+x^9)^{16} = \sum_{i=0}^{16}\binom{16}{i}x^{5i}(1+x^2+x^4)^i.$$

It is clear that if $i > 5$ or $i < 4$, then the coefficient of x^{29} in the expansion of $x^{5i}(1+x^2+x^4)^i$ is 0. Note also that if i is even, then the coefficient of x^{29} in the expansion of $x^{5i}(1+x^2+x^4)^i$ is also 0. Thus we only need to determine the coefficient of x^{29} in the expansion of $\binom{16}{i}x^{5i}(1+x^2+x^4)^i$ for $i = 5$.

When $i = 5$, we have

$$\begin{aligned}\binom{16}{i}x^{5i}(1+x^2+x^4)^i &= \binom{16}{5}x^{25}(1+x^2+x^4)^5 \\ &= \binom{16}{5}x^{25}\sum_{j=0}^{5}\binom{5}{j}(x^2+x^4)^j \\ &= \binom{16}{5}x^{25}\sum_{j=0}^{5}\binom{5}{j}x^{2j}(1+x^2)^j.\end{aligned}$$

It is clear that the coefficient of x^4 in the expansion of $\sum_{j=0}^{5}\binom{5}{j}x^{2j}(1+x^2)^j$ is $\binom{5}{1}+\binom{5}{2}=15$. Hence the answer is $\binom{16}{5}\times 15=65520$.

33. Answer: 401

For each n = 1, 2, 3, ..., since $d_n=\gcd(a_n,a_{n+1})$, we have $d_n \mid a_n$ and $d_n \mid a_{n+1}$. Thus $d_n \mid a_{n+1}-a_n$, that is, $d_n \mid (n+1)^2+100-(n^2+100)$, which gives $d_n \mid 2n+1$. Hence $d_n \mid 2(n^2+100)-n(2n+1)$, and we obtain $d_n \mid 200-n$. It follows that $d_n \mid 2(200-n)+2n+1$, that is, $d_n \mid 401$. Consequently, $1\le d_n \le 401$ for all positive integers n.

Now when n = 200, we have $a_n=a_{200}=200^2+100=401\times 100$ and $a_{n+1}=a_{201}=201^2+100=401\times 101$. Therefore $d_{200}=\gcd(a_{200},a_{201})=401$. Hence it follows that the maximum value of d_n when n ranges over all positive integers is 401, which is attained at n = 200.

34. Answer: 441

First we determine a_{2008} and a_{2009}. Suppose that $a_{2008}=\overline{x_1x_2x_3x_4x_5x_6x_7x_8}$, where the x_i's are distinct digits in $\{1,2,3,4,5,6,7,8\}$.

Let $A=\{a_k : k=1,2,\ldots,40320\}$.

Since 7! = 5040 > 2008, we deduce that $x_1=1$, as there are more than 2008 numbers in A such that the first digit is 1.

As 2 × 6! < 2008 < 3 × 6!, we have $x_2=4$, as there are less than 2008 numbers in A such that the first digit is 1 and the second digit is 2 or 3, but there are more than 2008 numbers in A such that the first digit is 1 and the second digit is 2, 3 or 4. Similarly, since 2 × 6! + 4 × 5! < 2008 < 2 × 6! + 5 × 5!, we see that the third digit x_3 is 7. By repeating the argument and using the inequalities

2 × 6! + 4 × 5! + 3 × 4! < 2008 < 2 × 6! + 4 × 5! + 4 × 4! and

2004 = 2 × 6! + 4 × 5! + 3 × 4! + 2 × 3! < 2008 < 2 × 6! + 4 × 5! + 3 × 4! + 3 × 3!,

we obtain $x_4=6$, $x_5=5$. Note also that among the numbers in A of the form 1476****, the digit 5 first appears as the fifth digit in a_{2005} if the numbers are arranged in increasing order. Consequently, as the last three digits are 2, 3 and 8, we must have $a_{2005}=14765238$. It follows that $a_{2006}=14765283$, $a_{2007}=14765328$, $a_{2008}=14765382$, and $a_{2009}=14765823$. Hence $a_{2009}-a_{2008}=14765823-14765382=441$.

35. Answer: 24

Write $u = \log_{10} x$. Then $\log_{10} \frac{100}{x} = 2 - u$. Since $a = \lfloor \log_{10} x \rfloor$, we have

$$u = a + \gamma \quad \text{for some } 0 \le \gamma < 1. \qquad (1)$$

Similarly, since $b = \lfloor 2 - u \rfloor$, we have

$$2 - u = b + \delta \quad \text{for some } 0 \le \delta < 1. \qquad (2)$$

Then $0 \le \gamma + \delta < 2$. Since $\gamma + \delta = u - a + (2 - u - b) = 2 - a - b$ is an integer, it follows that $\gamma + \delta = 0$ or $\gamma + \delta = 1$.

Case 1: $\gamma + \delta = 0$.

Then $\gamma = 0$ and $\delta = 0$, since $\gamma \ge 0$ and $\delta \ge 0$. Therefore

$$\begin{aligned} 2a^2 - 3b^2 &= 2u^2 - 3(2-u)^2 \\ &= -u^2 + 12u - 12 \\ &= 24 - (u-6)^2 \le 24, \end{aligned}$$

and the maximum value is attained when $u = 6$.

Case 2: $\gamma + \delta = 1$.

Then we must have $0 < \gamma, \delta < 1$ by (1) and (2). Also, by (1) and (2), we have $b = \lfloor 2 - u \rfloor = \lfloor 2 - a - \gamma \rfloor = 1 - a$. Thus

$$\begin{aligned} 2a^2 - 3b^2 &= 2a^2 - 3(1-a)^2 \\ &= -a^2 + 6a - 3 \\ &= 6 - (a-3)^2 \le 6. \end{aligned}$$

Hence the largest possible value of $2a^2 - 3b^2$ is 24, when $x = 10^6$.