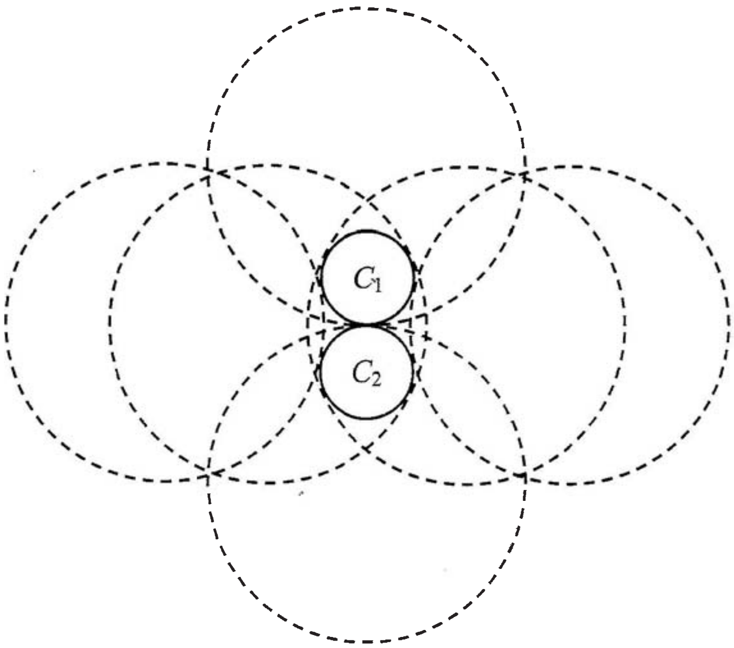
Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Junior Section Solutions)

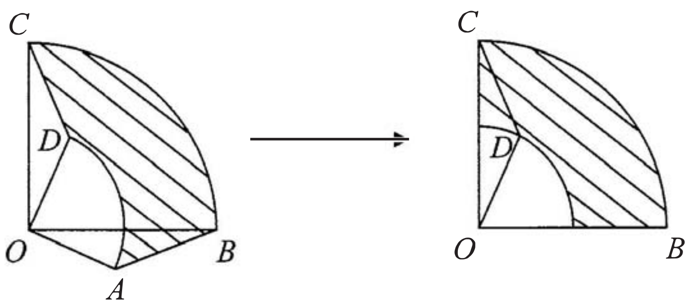
1 Answer: (C)

As seen from the diagram below, there are 6 circles that are tangent to both C_1 and C_2.



2 Answer: (A)

Rotate ΔOAB 90° anticlockwise about the point O to overlap with ΔODC.



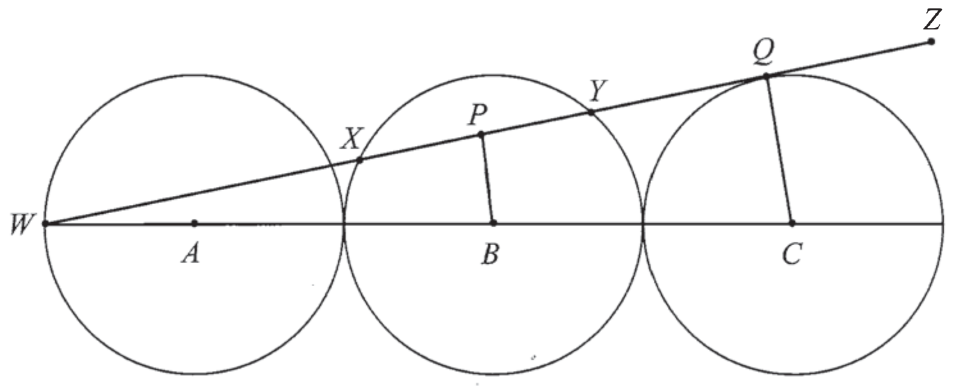
$\therefore$ Shaded area $= \frac{1}{4}(64\pi - 16\pi) = 12\pi$.

3 Answer: (D)

$\left(\sqrt{x-2}+\sqrt{7-x}\right)^2 = 5+2\sqrt{(x-2)(7-x)} = 5+2\sqrt{6.25-(x-4.5)^2}$.
Hence maximum value of $k=\sqrt{5+2(2.5)}=\sqrt{10}$.

4 Answer: (B)

Let P be the midpoint of XY and Q be the point where the line WZ meets the third circle.



Then by similar Δs, $\frac{PB}{60}=\frac{20}{100} \Rightarrow PB = 12$. Hence $XY = 2XP = 2\sqrt{20^2-12^2} = 32$.

5 Answer: (D)

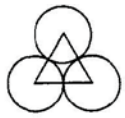
$y=\frac{10x}{10-x} \Rightarrow x = 10-\frac{100}{y+10} < 0$, so $\frac{100}{y+10} > 10 \Rightarrow -10 < y < 0$.
Since both x and y are integers, $y + 10$ is a factor of 100, i.e. $y = -9, -8, -6, -5$.
The maximum value of y is -5.

6 Answer: (C)

Let $b_n = a_n - a_{n-1} = n^2$. $\sum_{n=1}^{50} b_n = \sum_{n=1}^{50} n^2 = \frac{20}{100}(50)(51)(101) = 42925$.

On the other hand, $\sum_{n=1}^{50} b_n = a_{50} - a_0 \Rightarrow a_{50} = 42925 + a_0 = 44934$.

7 Answer: (B)



Consider the equilateral triangle formed by joining the centres of 3 adjacent coins. It is easy to see that the required percentage is given by the percentage of the triangle covered by these three coins.

We have $\dfrac{\frac{1}{2}\pi r^2}{\sqrt{3}\, r^2} \times 100\% = \dfrac{50}{\sqrt{3}}\,\pi\,\%$.

8 Answer: (E)

$x^2 + y^2 = 6 \Rightarrow (x + y)^2 = 2xy + 6$. So $2x + \{\, (x + y)^2 - 6 \,\} + 2y = 2(2 + 3\sqrt{2})$
$\Rightarrow\ (x + y)^2 + 2(x + y) + 1 = 11 + 6\sqrt{2} \Rightarrow (x + y + 1)^2 = (3 + \sqrt{2})^2$.

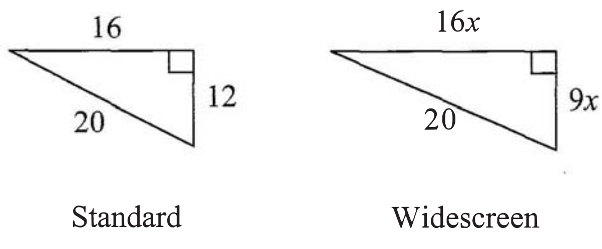
9 Answer: (E)

$y = (x^2 - 16^2)(x^2 - 14^2) = x^4 - 452x^2 + 50176 = (x^2 - 226)^2 - 900$.
So minimum value of $y = -900$.

10 Answer: (A)

$(a, b, c, d) = (2,3,7,42), (2,3,8,24), (2,3,9,18), (2,3,10,15), (2,4,5,20), (2,4,6,12)$.

11 Answer: (337)



$(16x)^2 + (9x)^2 = 20^2 \Rightarrow 337x^2 = 400$. $\therefore\ \dfrac{\text{Area of Standard}}{\text{Area of Widescreen}} = \dfrac{(16)(12)}{(16x)(9x)} = \dfrac{337}{300}$.

12 Answer: (47)

Let the area of the pentagon and the rectangle be P and R respectively.
We have $\frac{3}{16}P=\frac{2}{9}R$. So $\frac{m}{n}=\frac{13}{16}P\div\frac{7}{9}R=\frac{26}{21}\Rightarrow m+n=47$.

13 Answer: (513)

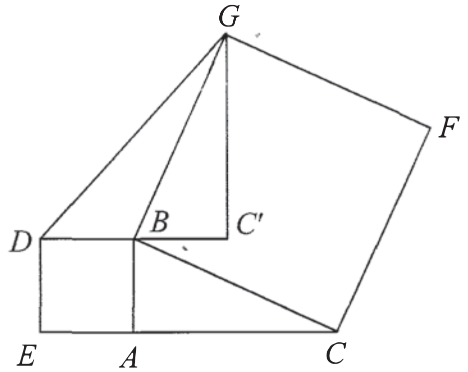
Minimum number of answer scripts is $2^9+1=513$.

14 Answer: (2520)

First we arrange the 6 girls in 6! ways. Next, there are 7 spaces between the 6 girls to insert the 5 boys. Hence $k={}^7C_5\times 5!=2520$.

15 Answer: (6)

We rotate ΔBAC 90° anticlockwise about the point B to get $\Delta BC'G$.
$C'G=BC=\sqrt{26-8}=\sqrt{18}$.



$\therefore$ The area of $\Delta DBG=\frac{1}{2}\times DB\times C'G=\frac{1}{2}\sqrt{8}\sqrt{18}=6\text{ cm}^2$.

16 Answer: (173)

Note that $\frac{1}{n\times(n+1)\times(n+2)}=\frac{1}{2}\left(\frac{1}{n\times(n+1)}-\frac{1}{(n+1)\times(n+2)}\right)$.
Hence $\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\frac{1}{4\times5\times6}+\ldots+\frac{1}{13\times14\times15}+\frac{1}{14\times15\times16}$
$=\frac{1}{2}\left(\frac{1}{2\times3}-\frac{1}{15\times16}\right)=\frac{13}{160}$.

17 Answer: (2)

Let $x = a + 1, y = b - 1\ (x - y \neq 0)$, then $x - 1 + \frac{1}{x} = y + 1 + \frac{1}{y} - 2 \Rightarrow x + \frac{1}{x} = y + \frac{1}{y}$

$\Rightarrow (x - y)\left(1 - \frac{1}{xy}\right) = 0 \Rightarrow xy = 1 \Rightarrow ab - a + b = 2.$

18 Answer: (2012)

Suppose $y \geq 0$. From $|y| - y + x = 7$ we have $x = 7$ and from $|x| + x + 5y = 2$ we have $y = -\frac{12}{5} < 0\ (\rightarrow\leftarrow)$. So $y < 0$.

Suppose $x \leq 0$. From $|x| + x + 5y = 2$ we have $y = \frac{2}{5} > 0\ (\rightarrow\leftarrow)$. So $x > 0$.

Hence the two equations become $-2y + x = 7$ and $2x + 5y = 2 \Rightarrow x = \frac{13}{3},\ y = -\frac{4}{3}$.

19 Answer: (7)

Since $3p + 3q = 6n - 27$, $p + q = 2n - 9$ which is odd. So $p = 2$, $q = 3$ and $n = 7$.

20 Answer: (6)

Note that $x\sqrt{y} + y\sqrt{x} + \sqrt{2009xy} - \sqrt{2009x} - \sqrt{2009y} - 2009$
$= (\sqrt{x} + \sqrt{y} - \sqrt{2009})(\sqrt{xy} - \sqrt{2009})$, so $xy = 2009$.
$\therefore (x, y) = (1, 2009), (7, 287), (41, 49), (49, 41), (287, 7), (2009, 1)$.

21 Answer: (286)

Let $L = \frac{1}{2003} + \frac{1}{2004} + \frac{1}{2005} + \frac{1}{2006} + \frac{1}{2007} + \frac{1}{2008} + \frac{1}{2009}$.

Clearly $\frac{7}{2009} < L < \frac{7}{2003} \Rightarrow 286\frac{1}{7} < \frac{1}{L} < 287$.

22 Answer: (26)

Looking at half of the area of rectangle *ABLJ*, we have $(ABK) = (ABD) + (DGJ)$
$\Rightarrow (ABC) + (BCEF) + (EFHI) + (HIK) = (ABC) + (ACD) + (DEIJ) + (EFHI) + (FGH)$
$\Rightarrow (BCEF) + (HIK) = (ACD) + (DEIJ) + (FGH)$
$\Rightarrow 500 + (HIK) = 22 + 482 + 22 \Rightarrow (HIK) = 26.$

23 Answer: (7)

Let $X = \sqrt[3]{77 - 20\sqrt{13}}$, $Y = \sqrt[3]{77 + 20\sqrt{13}}$ and $A = X + Y$.

Since $X^3 + Y^3 = 154, XY = \sqrt[3]{77^2 - 20^2 \times 13} = 9, A^3 = (X + Y)^3 = X^3 + Y^3 + 3XY(X + Y)$
$\Rightarrow A^3 = 154 + 27A \Rightarrow A^3 - 27A - 154 = 0 \Rightarrow (A - 7)(A^2 + 7A + 22) = 0 \Rightarrow A = 7$.

24 Answer: (133)

We consider 3 sets of integers:

(a) 0001 ~ 0999. We count the number of nonnegative solutions of $a + b + c = 11$. It is ${}^{11+3-1}C_{3-1} = {}^{13}C_2 = 78$. Note that 11 should not be split as 0+0+11 nor 0+1+10. So the number of solutions is $78 - 3 - 6 = 69$.

(b) 1001 ~ 1999. We count the number of nonnegative solutions of $a + b + c = 10$. It is ${}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$. Note that 10 should not be split as 0+0+10. So the number of solutions is $66 - 3 = 63$.

(c) 2001 ~ 2009. We see that only 2009 satisfies the property.

$\therefore$ The total number of integers satisfying the property is $69 + 63 + 1 = 133$.

25 Answer: (5)

Let $x = 1$, we have $a_0 + a_1 + a_2 + \ldots + a_n = 1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 3$.

Also, $a_1 = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$, $a_n = 1$,

so $60 - \dfrac{n(n+1)}{2} + \dfrac{n(n+1)}{2} + 1 = 2^{n+1} - 3 \Rightarrow n = 5$.

26 Answer: (26)

Note that area of $\Delta OPQ = \dfrac{1}{2} \times 4 \text{ cm} \times 13 \text{ cm} = 26 \text{ cm}^2$.

27 Answer: (7)

From given equation, $x_1 + x_2 + x_3 + x_4 = 0, = x_1 x_2 (x_3 + x_4) + x_3 x_4 (x_1 + x_2) = -90$ and $x_1 x_2 x_3 x_4 = -2009$. Now $x_1 x_2 = 49 \Rightarrow x_3 x_4 = -41$. So $x_1 + x_2 = 1, x_3 + x_4 = -1$. Hence $k = x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4 = 49 + (1) \times (-1) - 41 = 7$.

28 Answer: (1052)

Note that $(\text{Area of }\Delta OAB)^2 = (\text{Area of }\Delta OAC)^2 + (\text{Area of }\Delta OBC)^2 + (\text{Area of}\Delta ABC)^2$.
So $(\text{Area of }\Delta OAB)^2 + (\text{Area of }\Delta OAC)^2 + (\text{Area of }\Delta OBC)^2 + (\text{Area of }\Delta ABC)^2$
$= 2 \times \{(\text{Area of }\Delta OAC)^2 + (\text{Area of }\Delta OBC)^2 + (\text{Area of }\Delta ABC)^2\}$
$= 2 \times \{ (1/2 \times 7 \times 6)^2 + (1/2 \times 7 \times 2)^2 + (1/2 \times 2 \times 6)^2 \} = 1052$.

29 Answer: (111)

Consider $\dfrac{9n+11}{n-10} = 9 + \dfrac{101}{n-10}$. If $\dfrac{n-10}{9n+11}$ is a non-zero reducible fraction, then $\dfrac{101}{n-10}$ is also a non-zero reducible fraction $\Rightarrow$ Least positive integer $n = 111$.

30 Answer: (90)

$x^2 + 2(m+5)x + (100m+9) = 0 \Rightarrow x = -(m+5) \pm \sqrt{(m-45)^2 - 2009}$. This yields integer solutions if and only if $(m-45)^2 - 2009$ is a perfect square, say n^2.

Hence $(m-45)^2 - n^2 = 2009 = 7^2 \times 41 \Rightarrow$

$$|m-45| + n = 2009 \text{ and } |m-45| - n = 1, \text{ or}$$
$$|m-45| + n = 287 \text{ and } |m-45| - n = 7, \text{ or}$$
$$|m-45| + n = 49 \text{ and } |m-45| - n = 41.$$

Solving, $n = 45 \pm 1005, 45 \pm 147, 45 \pm 45 \Rightarrow$ The smallest positive n is 90.

31 Answer: (5)

Area of the triangle is $\frac{1}{2} \times AD \times BC = \frac{1}{2} \times AC \times BE = \frac{1}{2} \times AB \times CF$.

Since $AD = 4$ and $BE = 12$, $BC : AC = 3 : 1$. Let $AC = x$, then $BC = 3x$.

Using Triangle Inequality, $AB < AC + BC$ and $BC < AB + AC \Rightarrow 2x < AB < 4x$.

From $CF = \dfrac{12x}{AB}$, $3 < CF < 6 \Rightarrow$ The largest integer value for CF is 5.

32 Answer: (97)

We let the 2 distinct digits be A and B with $1 \leq A \leq 9$, $0 \leq B \leq 9$ and $A \neq B$.

Case 1: $ABAB = 101 \times AB$. There are $9 \times 9 = 81$ possibilities.
However AB must not be a multiple of 7 (12 possibilities excluding 07 and 77) $\Rightarrow$
There are $81 - 12 = 69$.

Case 2: $AABB = 11 \times (100A + B)$. There are 11 possibilities that are factors of 7.
$(A,B) = (1,5)$, $(2,3)$, $(3,1)$, $(3,8)$, $(4,6)$, $(5,4)$, $(6,2)$, $(6,9)$, $(7,0)$, $(8,5)$, $(9,3)$.

Case 3: $ABBA = 11 \times (91A + 10B)$. B must be 0 or 7. There are 17 possibilities.

$\therefore$ Total such possible number $= 69 + 11 + 17 = 97$.

33 Answer: (64)

If $n = 33$, $m = 1, 2, 3, \ldots, 32$, so there are 32 pairs.
If $m = 33$, $n = 33, 34, \ldots, 40$, so there are 8 pairs.

If n, $m \neq 33$, we have 2 cases:

Case 1: $m = 3a$, $n = 11b$ ($a \neq 11$ and $b \neq 3$), thus $1 \leq 3a \leq 11b \leq 40$.
So if $b = 1$, $a = 1, 2, 3$ and if $b = 2$, $a = 1, 2, 3, \ldots, 7$. There are 10 pairs.

Case 2: $m = 11a$, $n = 3b$ ($a \neq 3$ and $b \neq 11$), thus $1 \leq 11a \leq 3b \leq 40$.
So if $a = 1$, $b = 4, 5, 6, 7, 8, 9, 10, 12, 13$ and if $a = 2$, $b = 8, 9, 10, 12, 13$.
There are $9 + 5 = 14$ pairs.

Hence we have altogether $32 + 8 + 10 + 14 = 64$ pairs.

34 Answer: (3402)

We call a sequence that satisfy the condition a "good" sequence.

Let A_n denote the number of "good" sequence that end in either 0 or 4,
B_n denote the number of "good" sequence that end in either 1 or 3,
C_n denote the number of "good" sequence that end in 2.

We have

(1) $A_{n+1} = B_n$ because each sequence in A_{n+1} can be converted to a sequence in B_n by deleting its last digit.

(2) $B_{n+1} = A_n + 2C_n$ because each sequence in A_n can be converted into a sequence in B_{n+1} by adding a 1 (if it ends with a 0) or a 3 (if it ends with a 4) to its end, and each sequence in C_n can be converted into a sequence in B_n by adding a 1 or a 3 to it.

(3) $C_{n+1} = B_n$ because each sequence in C_{n+1} can be converted to a sequence in B_n by deleting the 2 at its end.

Hence we can show that $B_{n+1} = 3B_{n-1}$ for $n \geq 2$.

Check that $B_1 = 2$ and $B_2 = 4$, so $B_{2n+1} = 2 \times 3^n$ and $B_{2n} = 4 \times 3^{n-1}$.

So $A_{13} + B_{13} + C_{13} = 2B_{12} + B_{13} = 2 \times 4 \times 3^5 + 2 \times 3^6 = 3402$.

35 Answer: (79497)

Clearly, m and n are both 5-digit numbers.

Next, it would be helpful that we know $mn = 2 \times 3^5 \times 5 \times 7 \times 11^2 \times 19 \times 37$.

Now since the last digit of mn is 0, we may assume $5 \mid m$ and $2 \mid n$. But the first digit of mn is $1 \Rightarrow$ Last digit of m is 5 (not 0) and last digit of n is 2 (not 4, 6 or 8).

Also, $3^5 \mid mn$, so 9 divides at least one of m and n. On the other hand, $9 \mid m \Rightarrow 9 \mid n$. Similarly $11 \mid m \Rightarrow 11 \mid n$.

Set $n = 198k$. Then the last digit of k is 4 or 9.

Since the remaining factors 3, 7, 19, 37 are odd, the last digit of k must be 9.
We have only the following combinations: $k = 7 \times 37$ or $3 \times 7 \times 19$ or $3 \times 19 \times 37$.

Recall that the first digit of n is 5, so $50000 \leq 198k < 60000 \Rightarrow k = 7 \times 37$.

Hence $n = 198k = 51282$ and $m = 28215 \Rightarrow m + n = 79497$.