Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2022

Senior Section (Round 1)

Wednesday, 1 June 2022 **0930 – 1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Co-organizer
Department of Mathematics, NUS

Sponsored by
Micron

Multiple Choice Questions

1. Suppose the roots of $\frac{x^2}{2} + mx + n = 0$ are $\frac{m}{2}$ and $\frac{n}{3}$. Find the smallest value of mn.

   (A) -1080 (B) -90 (C) 0 (D) 90 (E) 1080

2. Which of the following is true?

   (A) $\sqrt[6]{\frac{1}{333}} < \sqrt[3]{\frac{1}{18}} < \sqrt{\frac{1}{7}}$ (B) $\sqrt[3]{\frac{1}{18}} < \sqrt[6]{\frac{1}{333}} < \sqrt{\frac{1}{7}}$ (C) $\sqrt[3]{\frac{1}{18}} < \sqrt{\frac{1}{7}} < \sqrt[6]{\frac{1}{333}}$

   (D) $\sqrt{\frac{1}{7}} < \sqrt[6]{\frac{1}{333}} < \sqrt[3]{\frac{1}{18}}$ (E) None of the above.

3. Suppose $\sqrt{(\log_{377\times 377} 2022)(\log_{377} 2022)} = \log_k 2022$. Find k.

   (A) $\sqrt{337}$ (B) $337^{\sqrt{2}}$ (C) $337\sqrt{2}$ (D) $\sqrt{337}^{\sqrt{2}}$ (E) $\sqrt{337 \times 2}$

4. Suppose $y = \cos^2 x - 7\cos x + 25$, where x is any real number. Find the range of y.

   (A) $17 \le y \le 33$ (B) $18 \le y \le 33$ (C) $19 \le y \le 33$ (D) $20 \le y \le 33$
   (E) None of the above

5. Suppose $\sin(180^\circ + x) = -\frac{7}{9}$, where $450^\circ < x < 540^\circ$. Find $\sin(2x)$.

   (A) $\frac{49}{81}\sqrt{2}$ (B) $\frac{56}{81}\sqrt{2}$ (C) $-\frac{56}{81}$ (D) $-\frac{49}{81}\sqrt{2}$ (E) $-\frac{56}{81}\sqrt{2}$

Short Questions

6. Find the value of

$$\left(\frac{\cos 10^\circ + \cos 50^\circ + \cos 70^\circ + \cos 110^\circ}{\cos 20^\circ}\right)^8.$$

7. Suppose $x^{20} + \frac{x^{10}}{2} - \frac{3^{2x}}{9} + \frac{1}{16} = 0$ for some positive real number x. Find the value of $4 \cdot 3^x - 12x^{10}$.

8. How many positive integers less than or equal to 2022 **cannot** be expressed as

$$\lfloor 2x + 1 \rfloor + \lfloor 5x + 1 \rfloor$$

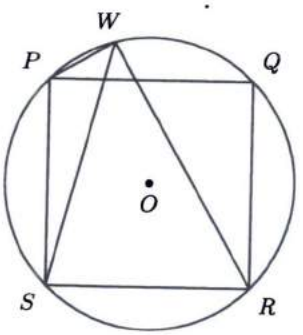
   for some real number x? Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. For example, $\lfloor -2.1 \rfloor = -3$, $\lfloor 3.9 \rfloor = 3$.

9. Suppose

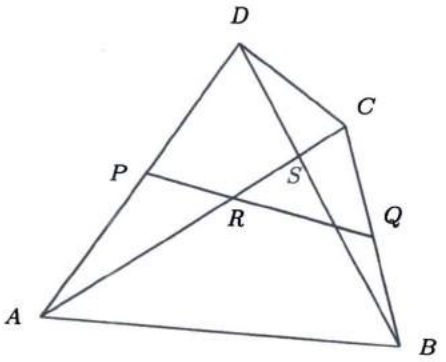
$$y = \frac{\tan^2 x - \tan x + \sqrt{33}}{\tan^2 x + \tan x + 1},$$

where $-90° < x < 90°$. Find the maximum possible value of $\sqrt{33}(y-5)$.

10. In the figure below, $PQRS$ is a square inscribed in a circle. Let W be a point on the arc PQ such that $WS = \sqrt{20}$. Find $(WP + WR)^2$.



11. The figure below shows a quadrilateral $ABCD$ such that $AC = BD$ and P and Q are the midpoints of the sides AD and BC respectively. The lines PQ and AC meet at R and the lines BD and AC meet at S. If $\angle PRC = 130°$, find the angle $\angle DSC$ (in °).



12. How many distinct terms are there if $(x^2+y^2)^{11}(x^{11}+y^{11})^9$ is algebraically expanded and simplified?

13. If $\sqrt{x^2+7x-4} + \sqrt{x^2-x+4} = x-1$, find the value of $3x^2+14x$.

14. Let $k = -1 + \sqrt{2022^{1/5} - 1}$, and let $f(x) = (k^2+2k+2)^{10x}$. Find the value of $\log_{2022} f(2022)$.

15. Find the smallest **odd** integer N, where $N > 2022$, such that when 1808, 2022 and N are each divided by a positive integer p, where $p > 1$, they all leave the same remainder.

16. If $\frac{12}{x}+\frac{48}{y}=1$, where x and y are positive real numbers, find the smallest possible value of $x+y$.

17. Find the largest value of $40x+60y$ if $x-y\leq 2$, $5x+y\geq 5$ and $5x+3y\leq 15$.

18. Suppose

$$\begin{aligned}\cos x-\cos y &= \frac{1}{2},\\ \sin x-\sin y &= -\frac{1}{3}.\end{aligned}$$

If $\sin(x+y)=\frac{m}{n}$, where $\frac{m}{n}$ is expressed as a fraction in its lowest terms, find the value of $m+n$.

19. For some positive integer n, the number n^3-3n^2+3n has a units digit of 6. Find the product of the last two digits of the number $7(n-1)^{12}+1$.

20. Find the largest positive integer n for which $\frac{20n+2020}{3n-6}$ is a positive integer.

21. In the xy-coordinate system, there are two circles passing through the point $(11,3\sqrt{3})$, and each of these circles is tangent to both the x-axis and the line $y=\sqrt{3}x$. Let S be the sum of the radii of the two circles. Find $\sqrt{3}S$.

22. Let P and Q be the points $(20(\sqrt{5}-1),0)$ and $(0,10(\sqrt{5}-1))$ on the xy-plane. Let R be the point (a,b). If $\angle PRQ$ is a right angle, find the maximum possible value of b.

23. How many positive integers n **do not** satisfy the inequality

$$n^{\frac{1}{3}\log_{20} n}>\sqrt{n}\ ?$$

24. Let $f(x)$ be a function such that

$$3f(x^2)+f(13-4x)=3x^2-4x+293$$

for all real number x. Find the value of $f(1)$.

25. Find the largest positive integer M such that

$$\cos^2 x-\sin^2 x+\sin x=\frac{M}{888}$$

has a real solution.