Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

Junior Section (First Round)

Tuesday, 4 June 2013 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. If $a = 8^{53}$, $b = 16^{41}$ and $c = 64^{27}$, then which of the following inequalities is true?

   (A) $a > b > c$ (B) $c > b > a$ (C) $b > a > c$ (D) $b > c > a$ (E) $c > a > b$

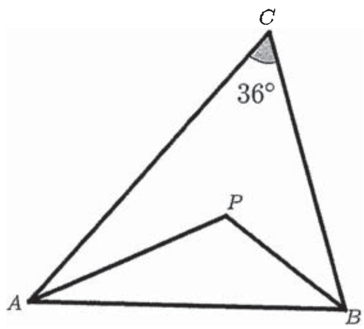
2. If a, b, c are real numbers such that $|a - b| = 1$, $|b - c| = 1$, $|c - a| = 2$ and $abc = 60$, find the value of $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}$.

   (A) $\frac{1}{30}$ (B) $\frac{1}{20}$ (C) $\frac{1}{10}$ (D) $\frac{1}{4}$ (E) None of the above

3. If x is a complex number satisfying $x^2 + x + 1 = 0$, what is the value of $x^{49} + x^{50} + x^{51} + x^{52} + x^{53}$?

   (A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

4. In $\triangle ABC$, $\angle ACB = 36°$ and the interior angle bisectors of $\angle CAB$ and $\angle ABC$ intersect at P. Find $\angle APB$.



   (A) 72° (B) 108° (C) 126° (D) 136° (E) None of the above

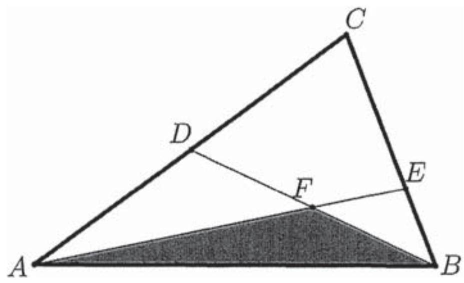
5. Find the number of integer pairs x, y such that $xy - 3x + 5y = 0$.

   (A) 1 (B) 2 (C) 4 (D) 8 (E) 16

6. Five young ladies were seated around a circular table. Miss Ong was sitting between Miss Lim and Miss Mak. Ellie was sitting between Cindy and Miss Nai. Miss Lim was between Ellie and Amy. Lastly, Beatrice was seated with Miss Poh on her left and Miss Mak on her right. What is Daisy's surname?

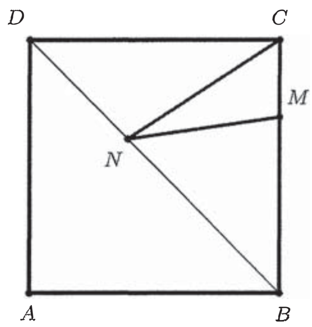
   (A) Lim (B) Mak (C) Nai (D) Ong (E) Poh

7. Given that ABC is a triangle with D being the midpoint of AC and E a point on CB such that $CE = 2EB$. If AE and BD intersect at point F and the area of $\triangle AFB = 1$ unit, find the area of $\triangle ABC$.



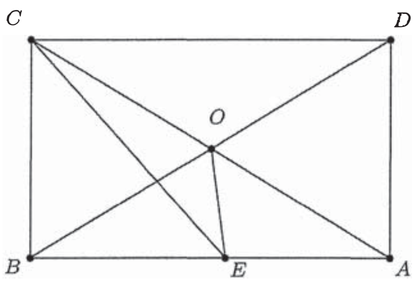
(A) 3 (B) $\frac{10}{3}$ (C) $\frac{11}{3}$ (D) 4 (E) 5

8. $ABCD$ is a square with sides 8 cm. M is a point on CB such that $CM = 2$ cm. If N is a variable point on the diagonal DB, find the least value of $CN + MN$.



(A) 8 (B) $6\sqrt{2}$ (C) 10 (D) $8\sqrt{2}$ (E) 12

9. $ABCD$ is a rectangle whose diagonals intersect at point O. E is a point on AB such that CE bisects $\angle BCD$. If $\angle ACE = 15°$, find $\angle BOE$.



(A) 60° (B) 65° (C) 70° (D) 75° (E) 80°

10. Let S be the smallest positive multiple of 15, that comprises exactly $3k$ digits with k '0's, k '3's and k '8's. Find the remainder when S is divided by 11.

(A) 0 (B) 3 (C) 5 (D) 6 (E) 8

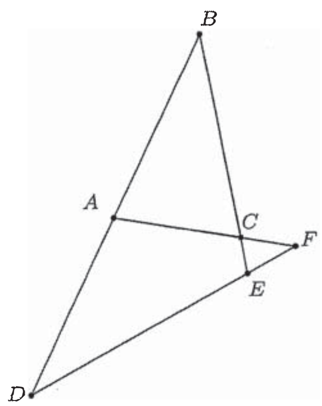
Short Questions

11. Find the value of $\sqrt{9999^2 + 19999}$.

12. If the graphs of $y = x^2 + 2ax + 6b$ and $y = x^2 + 2bx + 6a$ intersect at only one point in the xy-plane, what is the x-coordinate of the point of intersection?

13. Find the number of multiples of 11 in the sequence $99, 100, 101, 102, \cdots, 20130$.

14. In the figure below, BAD, BCE, ACF and DEF are straight lines. It is given that $BA = BC$, $AD = AF$, $EB = ED$. If $\angle BED = x°$, find the value of x.

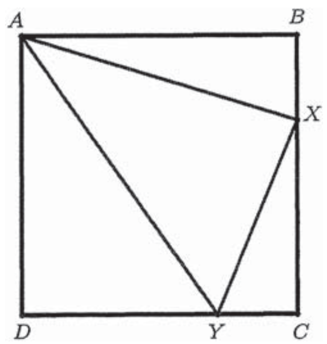


15. If $a = 1.69$, $b = 1.73$ and $c = 0.48$, find the value of

$$\frac{1}{a^2 - ac - ab + bc} + \frac{2}{b^2 - ab - bc + ac} + \frac{1}{c^2 - ac - bc + ab}.$$

16. Suppose that x_1 and x_2 are the two roots of the equation $(x - 2)^2 = 3(x + 5)$. What is the value of the expression $x_1 x_2 + x_1^2 + x_2^2$?

17. Let $ABCD$ be a square and X and Y be points such that the lengths of XY, AX and AY are 6, 8 and 10 respectively. The area of $ABCD$ can be expressed as $\frac{m}{n}$ units where m and n are positive integers without common factors. Find the value of $m + n$.



18. Let x and y be real numbers satisfying the inequality

$$5x^2 + y^2 - 4xy + 24 \leq 10x - 1.$$

Find the value of $x^2 + y^2$.

19. A painting job can be completed by Team A alone in 2.5 hours or by Team B alone in 75 minutes. On one occasion, after Team A had completed a fraction $\frac{m}{n}$ of the job, Team B took over immediately. The whole painting job was completed in 1.5 hours. If m and n are positive integers with no common factors, find the value of $m + n$.

20. Let a, b and c be real numbers such that $\dfrac{ab}{a+b} = \dfrac{1}{3}, \dfrac{bc}{b+c} = \dfrac{1}{4}$ and $\dfrac{ca}{c+a} = \dfrac{1}{5}$. Find the value of $\dfrac{24abc}{ab+bc+ca}$.

21. Let x_1 and x_2 be two real numbers that satisfy $x_1 x_2 = 2013$. What is the minimum value of $(x_1 + x_2)^2$?

22. Find the value of $\sqrt{45 - \sqrt{2000}} + \sqrt{45 + \sqrt{2000}}$.

23. Find the smallest positive integer k such that $(k - 10)^{4026} \geq 2013^{2013}$.

24. Let a and b be two real numbers. If the equation $ax + (b - 3) = (5a - 1)x + 3b$ has more than one solution, what is the value of $100a + 4b$?

25. Let $S = \{1, 2, 3, \ldots, 48, 49\}$. What is the maximum value of n such that it is possible to select n numbers from S and arrange them in a circle in such a way that the product of any two adjacent numbers in the circle is less than 100?

26. Given any 4-digit positive integer x not ending in '0', we can reverse the digits to obtain another 4-digit integer y. For example if x is 1234 then y is 4321. How many possible 4-digit integers x are there if $y - x = 3177$?

27. Find the least positive integer n such that $2^8 + 2^{11} + 2^n$ is a perfect square.

28. How many 4-digit positive multiples of 4 can be formed from the digits $0, 1, 2, 3, 4, 5, 6$ such that each digit appears without repetition?

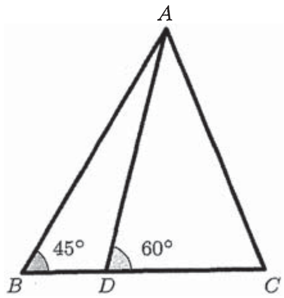
29. Let m and n be two positive integers that satisfy

$$\frac{m}{n} = \frac{1}{10 \times 12} + \frac{1}{12 \times 14} + \frac{1}{14 \times 16} + \cdots + \frac{1}{2012 \times 2014}.$$

Find the smallest possible value of $m + n$.

30. Find the units digit of $2013^1 + 2013^2 + 2013^3 + \cdots + 2013^{2013}$.

31. In $\triangle ABC$, $DC = 2BD$, $\angle ABC = 45°$ and $\angle ADC = 60°$. Find $\angle ACB$ in degrees.



32. If a and b are positive integers such that $a^2 + 2ab - 3b^2 - 41 = 0$, find the value of $a^2 + b^2$.

33. Evaluate the following sum

$$\left\lfloor \frac{1}{1} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor + \left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{3}{3} \right\rfloor + \left\lfloor \frac{1}{4} \right\rfloor + \left\lfloor \frac{2}{4} \right\rfloor + \left\lfloor \frac{3}{4} \right\rfloor + \left\lfloor \frac{4}{4} \right\rfloor + \left\lfloor \frac{1}{5} \right\rfloor + \cdots,$$

up to the 2013^{th} term.

34. What is the smallest possible integer value of n such that the following statement is always true?

In any group of $2n - 10$ *persons, there are always at least* 10 *persons who have the same birthdays.*

(For this question, you may assume that there are exactly 365 different possible birthdays.)

35. What is the smallest positive integer n, where $n \neq 11$, such that the highest common factor of $n - 11$ and $3n + 20$ is greater than 1?