Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Open Section, Round 2)

Saturday, 6 July 2013   0900-1300

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1. Let $a_1, a_2, \ldots$ be a sequence of integers defined recursively by $a_1 = 2013$ and for $n \geq 1$, a_{n+1} is the sum of the 2013th power of the digits of a_n. Do there exist distinct positive integers i, j such that $a_i = a_j$?

2. Let ABC be an acute-angled triangle and let D, E and F be the midpoints of BC, CA and AB respectively. Construct a circle, centred at the orthocentre of triangle ABC, such that triangle ABC lies in the interior of the circle. Extend EF to intersect the circle at P, FD to intersect the circle at Q and DE to intersect the circle at R. Show that $AP = BQ = CR$.

3. Let N be a positive integer. Prove that there exists a positive integer n such that $n^{2013} - n^{20} + n^{13} - 2013$ has at least N distinct prime factors.

4. Let F be a finite nonempty set of integers and let n be a positive integer. Suppose that

   - Any $x \in F$ may be written as $x = y + z$ for some $y, z \in F$;

   - If $1 \leq k \leq n$ and $x_1, \ldots, x_k \in F$, then $x_1 + \cdots + x_k \neq 0$.

   Show that F has at least $2n + 2$ distinct elements.

5. Let ABC be a triangle with integral side lengths such that $\angle A = 3\angle B$. Find the minimum value of its perimeter.