Solutions

1. Answer: (D).

   The original values are $2100 \div 120\% = 1750$ and $2100 \div 80\% = 2625$ respectively. Then the profit is

   $$1750 + 2625 - 2 \times 2100 = -175.$$

2. Answer: (D).

   If $x + 2013 = 0$, then $x = -2013$. Suppose $x + 2013 \neq 0$. Then $x^3 - x - 1 = \pm 1$.

   If $x^3 - x - 1 = 1$, there is no integer solution; if $x^3 - x - 1 = -1$, then $x = 0, 1, -1$. Since $x + 2013$ is even, $x = 1$ or $x = -1$.

3. Answer: (E).

   (A) and (B) are the reflections with respect to $y = x$ and $y = -x$ respectively; (C) and (D) are the rotations about the origin by $90°$ and $-90°$ respectively.

4. Answer: (A).

   For any positive integer n,

   $$\begin{aligned} 2013^n - 1803^n - 1781^n + 1774^n &= (2013^n - 1803^n) - (1781^n - 1774^n) \\ &= (2013 - 1803)u - (1781 - 1774)v = 210u - 7v, \\ 2013^n - 1803^n - 1781^n + 1774^n &= (2013^n - 1781^n) - (1803^n - 1774^n) \\ &= (2013 - 1781)x - (1803 - 1774)y = 29x - 29y. \end{aligned}$$

   So $2013^n - 1803^n - 1781^n + 1774^n$ is divisible by $7 \times 29 = 203$ for every positive integer n.

5. Answer: (C).

   Rewrite the equation as

   $$xy^2 + y^2 - x - y = (y - 1)(x(y + 1) + y) = n.$$

   If $n = 0$, then there are infinitely many integer solutions. Suppose $n \neq 0$, and the equation has infinitely many integer solutions. Then there exists a divisor k of n such that $y - 1 = k$ and $x(y + 1) + y = n/k$ for infinitely many x. It forces $y + 1 = 0$, i.e., $y = -1$. Then $n = 2$.

6. Answer: (E).

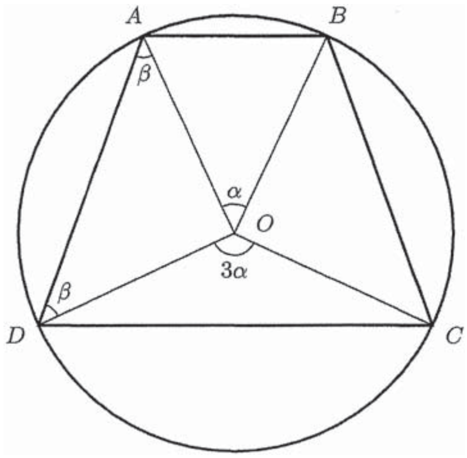
Let $k = \sin\theta\cos\theta$. Then

$$\frac{13}{36} = \left(\frac{1}{\sin\theta} - \frac{1}{\cos\theta}\right)^2 = \frac{1 - 2\sin\theta\cos\theta}{(\sin\theta\cos\theta)^2} = \frac{1-2k}{k^2}.$$

Solve the equation: $k = 6/13$ ($k = -6$ is rejected). Then $\sin 2\theta = 2k = 12/13$ and

$$\cot\theta - \tan\theta = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta} = \frac{5/13}{6/13} = \frac{5}{6}.$$

7. Answer: (A).



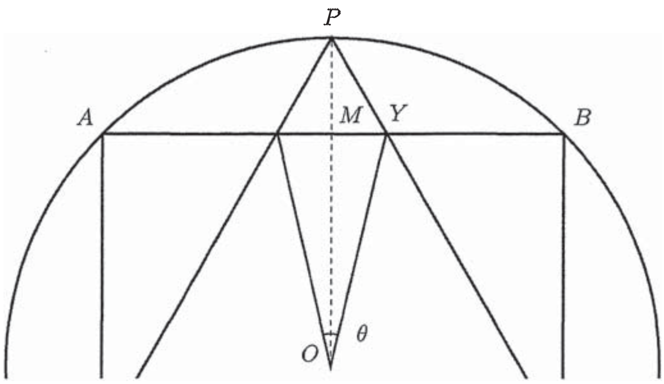
Let $\angle AOB = \alpha$ and $\angle ADO = \beta$. Then $\frac{1}{2}(\alpha + 3\alpha) = 2\beta$; that is, $\alpha = \beta$. Given that

$$\frac{5}{2} = \frac{\sin(3\alpha/2)}{\sin(\alpha/2)} = \frac{3\sin(\alpha/2) - 4\sin^3(\alpha/2)}{\sin(\alpha/2)} = 3 - 4\sin^2(\alpha/2).$$

Then $\sin^2(\alpha/2) = \frac{1}{8}$. Hence,

$$\frac{S_{\triangle BOC}}{S_{\triangle AOB}} = \frac{\sin(\pi - 2\alpha)}{\sin\alpha} = 2\cos\alpha = 2\left(1 - 2\sin^2(\alpha/2)\right) = \frac{3}{2}.$$

8. Answer: (C).



Let the side of the square be 2. Then the radius of the circle is $\sqrt{2}$. Let $\theta = \angle XOY$. So

$$\tan(\theta/2) = \frac{MY}{MO} = MY = PM \tan 30^\circ = \frac{\sqrt{2}-1}{\sqrt{3}}.$$

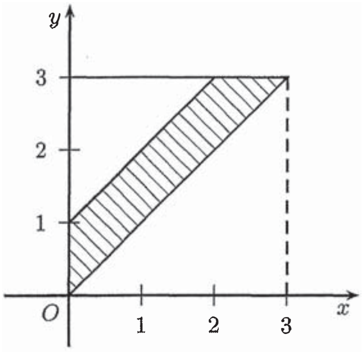
Then

$$\tan\theta = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} = \frac{\sqrt{6}-\sqrt{3}}{\sqrt{2}}.$$

9. Answer: (E).

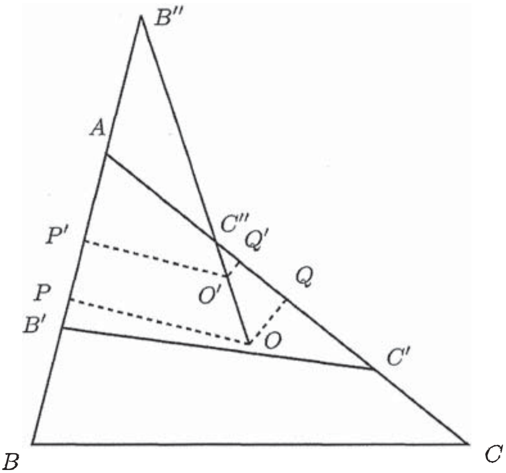
Let x and y denote the numbers of hours after 2:00p.m. that the first and the second person visits the swimming pool, respectively. Then

$$0 \le x \le y \le 3 \quad \text{and} \quad y \le x+1.$$



So the chance that they meet is $\dfrac{5/2}{9/2} = \dfrac{5}{9}$.

10. Answer: (D).



Let P and P' be the projections of O and O' on AB respectively. Then

$$PP' = AP - AP' = \frac{1}{2}AB - \frac{1}{2}AB' = \frac{1}{2}(AB - AB') = \frac{1}{2}BB'.$$

Similarly, let Q and Q' be the projections of O and O' on AC respectively, then $QQ' = \frac{1}{2}CC'$.

$$\sin\angle O'OP = \frac{PP'}{OO'} = \frac{QQ'}{OO'} = \sin\angle O'OQ \Rightarrow \angle O'OP = \angle O'OQ.$$

So $\angle AB''C'' = \angle AC''B''$. It follows that $AB'' = AC''$.

11. Answer: 8000.

$\tan\alpha = 4\tan\beta = \frac{4}{\tan\alpha} \Rightarrow \tan\alpha = 2$. The two legs are $\frac{200}{\sqrt{5}}$ and $\frac{400}{\sqrt{5}}$ respectively.

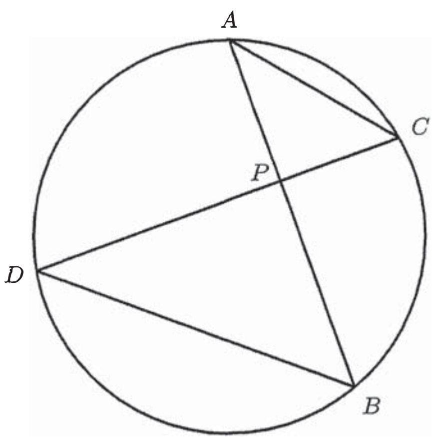
$$\text{Area} = \frac{1}{2} \times \frac{200}{\sqrt{5}} \times \frac{400}{\sqrt{5}} = 8000.$$

12. Answer: 595.

Let $f(x) = \frac{1+10x}{10-100x}$. Then $f^2(x) = -\frac{1}{100x}$, $f^3(x) = \frac{1-10x}{10+100x}$ and $f^4(x) = x$. Then

$$\begin{aligned} f\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + \cdots + f^{6000}\left(\frac{1}{2}\right) &= 1500\left[f\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + f^4\left(\frac{1}{2}\right)\right] \\ &= 1500\left(-\frac{3}{10} - \frac{1}{50} + \frac{1}{15} + \frac{1}{2}\right) = 595. \end{aligned}$$

13. Answer: 6.



Since $\angle A = \angle D$ and $\angle C = \angle B$, the triangles $\triangle ACP$ and $\triangle DBP$ are similar. Then

$$\frac{DP}{BP} = \frac{AP}{CP} \Rightarrow DP = \frac{AP}{CP} \times BP = \frac{2}{1} \times 3 = 6.$$

14. Answer: 80.

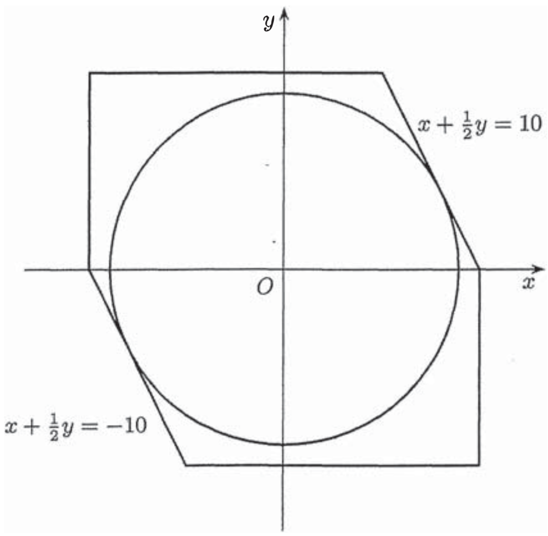
The region S is the hexagon enclosed by the lines

$$x = \pm 10, \quad y = \pm 10, \quad x + \frac{1}{2}y = \pm 10.$$

The largest circle contained in S is tangent to $x+\frac{1}{2}y=\pm 10$. Hence, its radius is the distance from the origin $(0,0)$ to $x+\frac{1}{2}y=10$:

$$r=\frac{10}{\sqrt{1+(1/2)^2}}=4\sqrt{5}.$$

The area of the largest circle is thus $\pi r^2=\pi(4\sqrt{5})^2=80\pi$.



15. Answer: 10.

Let $x_1=\sqrt[3]{17-\frac{27}{4}\sqrt{6}}$ and $x_2=\sqrt[3]{17+\frac{27}{4}\sqrt{6}}$. Then

$$b=x_1x_2=\sqrt[3]{\left(17-\frac{27}{4}\sqrt{6}\right)\left(17+\frac{27}{4}\sqrt{6}\right)}=\sqrt[3]{\frac{125}{8}}=\frac{5}{2},$$

$$\begin{aligned}a^3&=(x_1+x_2)^3=x_1^3+x_2^3+3x_1x_2(x_1+x_2)\\&=\left(17-\frac{27}{4}\sqrt{6}\right)+\left(17+\frac{27}{4}\sqrt{6}\right)+3ab=34+\frac{15}{2}a.\end{aligned}$$

Then $a=4$ and thus $ab=10$.

16. Answer: 101.

Case 1: $n<1000$. Write $n=\overline{abc}$. Then

$$a+b+c=9,\qquad a,b,c\in\{0,1,\ldots,9\}.$$

Case 2: $1000\le n<2000$. Write $n=\overline{1abc}$. Then

$$a+b+c=8,\qquad a,b,c\in\{0,1,\ldots,8\}.$$

Case 3: $2000\le n\le 2013$. Then $n=2007$.

Therefore, there are $\binom{9+3-1}{9}+\binom{8+3-1}{8}+1=55+45+1=101$ such numbers.

17. Answer: 10001.

Let $n = 0$. Since $p(m) - 1 \mid m^2$ for all m, $\deg p(x) \leq 2$. Let $p(x) = ax^2 + bx + c$. Then

$$\frac{p(m) - p(n)}{m - n} = \frac{a(m^2 - n^2) + b(m - n)}{m - n} = a(m + n) + b$$

divides $m + n$. If $a \neq 0$, then $a = \pm 1$ and $b = 0$; if $a = 0$, then $b = \pm 1$. Thus

$$p(x) = \pm x^2 + c, \ \pm x + c, \ c.$$

Since $p(0) = 1$ and $p(1) = 2$, we have $p(x) = x^2 + 1$ or $p(x) = x + 1$. The largest possible value of $p(100)$ is $100^2 + 1 = 10001$.

18. Answer: 31.

Let $k = \dfrac{b^3 - a^3}{a^2 b} = \left(\dfrac{b}{a}\right)^2 - \dfrac{a}{b}$. Then

$$\left(\frac{a}{b}\right)^3 + k\left(\frac{a}{b}\right)^2 - 1 = 0.$$

The only possible positive rational solution of $x^3 + kx^2 - 1 = 0$ is $x = 1$; namely, $a = b$. Conversely, if $a = b$, then it is obvious that $a^2 b \mid (b^3 - a^3)$.

Then $2013 > a^2 + b^2 = 2a^2$ implies $a \leq 31$.

19. Answer: 0.

Let $y = -x$. Then $g(f(0)) = f(x)$ for all x. This shows that f is a constant function; namely $f(x) = c$ for some c. So that $g(c) = g(f(0)) = f(x) = c$. For all x, y, we have

$$(x + y)g(y) = g(f(x + y)) - f(x) = g(c) - c = 0.$$

Since $x + y$ is arbitrary, we must have $g(y) = 0$ for all y. Hence,

$$g(0) + g(1) + \cdots + g(2013) = 0.$$

20. Answer: 36.

Let x, y and z denote the numbers of chocolate, licorice stick and lolly, respectively. Then

$$x + 0.5y + 0.4z = 10.$$

For each $k = 0, \ldots, 10$, consider $0.5y + 0.4z = k$, i.e., $5y + 4z = 10k$. Then

$$2 \mid y \quad \text{and} \quad 5 \mid z.$$

Set $y = 2s$ and $z = 5t$. Then $10s + 20t = 10k$, i.e., $s + 2t = k$. Then $t = 0, \ldots, \lfloor k/2 \rfloor$. So there are $\lfloor k/2 \rfloor + 1$ ways to use k dollars. The total number of ways is

$$\sum_{k=0}^{10} (\lfloor k/2 \rfloor + 1) = 1 + 1 + 2 + 2 + \cdots + 5 + 5 + 6 = 36.$$

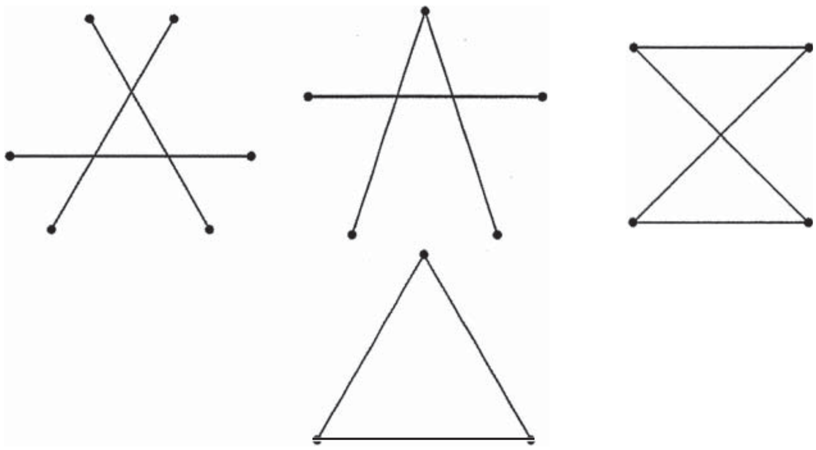
21. Answer: 81.

Suppose $f(f(f(n))) = n$ for all n. For some $k \in \{0, 1, 2\}$, there exist distinct a_i, b_i, c_i, $i \leq k$, such that $f(a_i) = b_i$, $f(b_i) = c_i$ and $f(c_i) = a_i$ and $f(n) = n$ if $n \neq a_i, b_i, c_i$. So the total number of required functions is

$$\binom{6}{6} + \binom{6}{3} \times 2 + \frac{1}{2} \times \binom{6}{3}\binom{3}{3} \times 2^2 = 81.$$

22. Answer: 287.

A triangle can be formed using 3, 4, 5 or 6 vertices.



So the total number is

$$\binom{7}{6} + 5 \times \binom{7}{5} + 2 \times 2 \times \binom{7}{4} + \binom{7}{3} = 287.$$

23. Answer: 66.

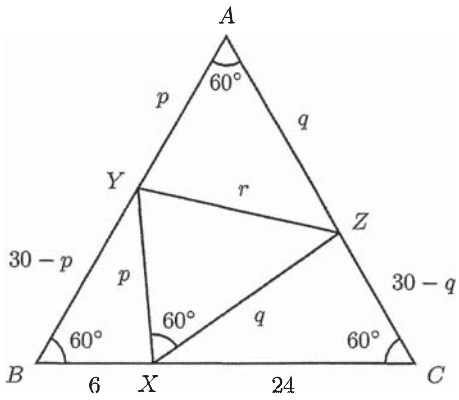
Let $n \geq 2$ be an integer, and let S_n denote the number of ways to paint n seats $a_1, \ldots, a_n$ as described, but with a_1 painted red. Consider S_{n+2} where $n \geq 2$.

Case 1: a_3 is painted red. Then there are 2 choices for a_2. Thus, the total number of ways for this case is $2S_n$.

Case 2: a_3 is not painted red. Since the colour of a_2 is uniquely determined by the colour of a_3, this is equivalent to the case when there are $(n + 1)$ seats. The total number of ways for this case is S_{n+1}.

We conclude that $S_{n+2} = S_{n+1} + 2S_n$. It is clear that $S_2 = S_3 = 2$. Then $S_4 = 6$, $S_5 = 10$ and $S_6 = 22$. So the required number of ways is $3 \times 22 = 66$.

24. Answer: 343.



Apply the law of cosine on $\triangle XBY$ and $\triangle XCZ$ respectively:

$$p^2 = 6^2 + (30 - p)^2 - 6(30 - p),$$
$$q^2 = 24^2 + (30 - q)^2 - 24(30 - q).$$

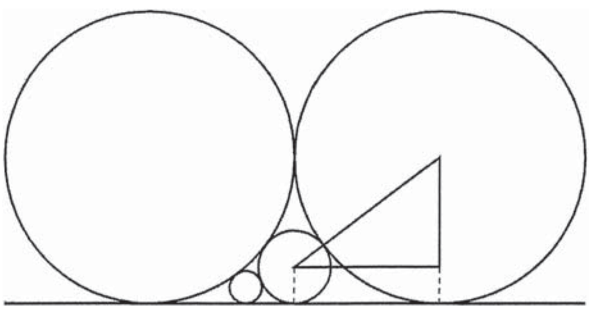
Then $p = 14$ and $q = 21$. Applying the law of cosine in $\triangle YXZ$ again to obtain

$$k = r^2 = p^2 + q^2 - pq = 14^2 + 21^2 - 14 \cdot 21 = 343.$$

25. Answer: 40.

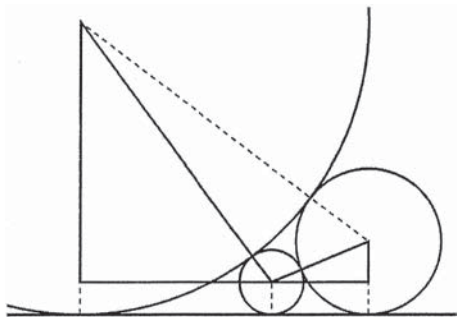
Let R be the radius of C_3. Then

$$(360 - R)^2 + 360^2 = (360 + R)^2 \Rightarrow R = 90.$$



Let r be the radius of C_4. Then

$$\sqrt{(360 + r)^2 - (360 - r)^2} + \sqrt{(90 + r)^2 - (90 - r)^2} = 360 \Rightarrow r = 40.$$



26. Answer: 181.

Since $\{x\}+\{x^2\}=1$, $x+x^2=n$ for some integer n. Then

$$x=\frac{-1\pm\sqrt{1+4n}}{2}.$$

$\frac{-1+\sqrt{1+4n}}{2}\le 10$ gives $0\le n\le 110$; $\frac{-1-\sqrt{1+4n}}{2}\ge -10$ implies $0\le n\le 90$.

If $\{x\}+\{x^2\}\neq 1$, then $\{x\}+\{x^2\}=0$, which happens only if x is an integer between -10 to 10. So the total number of solutions to $\{x\}+\{x^2\}=1$ is $111+91-21=181$.

27. Answer: 2040.

Let $\alpha=\frac{3+\sqrt{17}}{2}$ and $\beta=\frac{3-\sqrt{17}}{2}$. Then $\alpha\beta=-2$ and $\alpha+\beta=3$.

Set $S_n=\alpha^n+\beta^n$. Then

$$\begin{aligned}3S_{n+1}+2S_n&=(\alpha+\beta)(\alpha^{n+1}+\beta^{n+2})-\alpha\beta(\alpha^n+\beta^n)\\&=\alpha^{n+2}-\beta^{n+2}=S_{n+2}.\end{aligned}$$

Note that $|\beta|<1$. Then for even positive integer n, $\lfloor\alpha^n\rfloor=S_n+\lfloor-\beta^n\rfloor=S_n-1$.

Since $S_0=2$ and $S_1=3$, we can proceed to evaluate that $S_6=2041$.

28. Answer: 300.

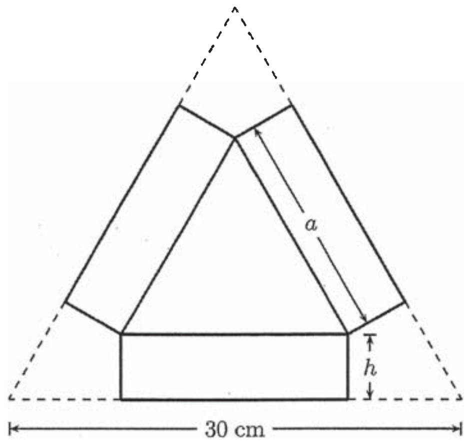
The area is $12\times\frac{1}{2}\times 10^2\times\sin 30^\circ=300$.

29. Answer: 500.

Let the length and the height of the box be a and h, respectively. Note that $a+2\sqrt{3}\,h=30$. Then the volume of the box is

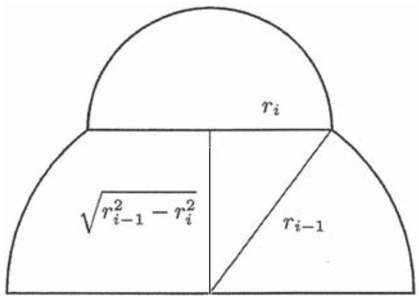
$$\frac{\sqrt{3}\,a^2}{4}h=\frac{1}{2}\left(\frac{a}{2}\cdot\frac{a}{2}\cdot 2\sqrt{3}\,h\right)\le\frac{1}{2}\left(\frac{a+2\sqrt{3}h}{3}\right)^3=\frac{1}{2}\times 10^3=500.$$

The equality holds if $a/2=2\sqrt{3}h$, i.e., $a=20$ and $h=5/\sqrt{3}$.



30. Answer: 300.

Let the radius of the ith hemisphere be r_i metre ($r_0 = 1$). Set $h_i = \sqrt{r_{i-1}^2 - r_i^2}$.



By Cauchy inequality,

$$\left(\sqrt{r_0^2 - r_1^2} + \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + r_3\right)^2 \leq 4(r_0^2 - r_1^2 + r_1^2 - r_2^2 + r_2^2 - r_3^2 + r_3^2)$$
$$= 4r_0^2 = 4.$$

The total height $h \leq r_0 + \sqrt{r_0^2 - r_1^2} + \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + r_3 \leq 1 + \sqrt{4} = 3\,\text{m} = 300\,\text{cm}$.

The equality holds if $r_i = \dfrac{\sqrt{4-i}}{2}$, $i = 0, 1, 2, 3$.

31. Answer: 5.

It is clear that $|x|, |y|, |z| \leq 1$. Note that

$$0 = x + y + z - x^2 - y^2 - z^2 = x(1-x) + y(1-y) + z(1-z).$$

Without loss of generality, assume that $z \leq 0$. Then $x = (1-y) + (-z) \geq 0$ and similarly $y \geq 0$. Since

$$1 - z^2 = x^2 + y^2 \geq \frac{(x+y)^2}{2} = \frac{(1-z)^2}{2},$$

we have $-1/3 \le z \le 0$. On the other hand,

$$\begin{aligned} x^3+y^3+z^3 &= (x+y)\left[\frac{3(x^2+y^2)-(x+y)^2}{2}\right]+z^3 \\ &= (1-z)\left[\frac{3(1-z^2)-(1-z)^2}{2}\right]+z^3 = 1-3z^2+3z^3 \end{aligned}$$

increases as z increases. The minimum $m = 5/9$ is obtained at $z = -1/3$ and $x = y = 2/3$.

32. Answer: 4800.

Let S_n denote the area of the region obtained in the nth step. Then $S_0 = 25\sqrt{3}$ and $S_n - S_{n-1} = \frac{3}{4}\left(\frac{4}{9}\right)^n S_0$ for all $n \ge 1$. Then

$$\begin{aligned} S_n &= S_0 + \frac{3}{4}\left[\frac{4}{9}+\left(\frac{4}{9}\right)^2+\cdots+\left(\frac{4}{9}\right)^n\right]S_0 \\ &= S_0\left[1+\frac{3}{4}\cdot\frac{4}{9}\cdot\frac{1-(4/9)^n}{1-4/9}\right] = 25\sqrt{3}\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right]. \end{aligned}$$

As n increases, S_n tends to $S = 25\sqrt{3}\cdot\frac{8}{5} = 40\sqrt{3}$. So $S^2 = 4800$.

33. Answer: 60024.

Let $x = \frac{1}{2013^{1000}}$. Then

$$\begin{aligned} xn! &= a_1 n! + a_2\frac{n!}{2!}+\cdots+a_{n-1}\frac{n!}{(n-1)!}+a_n\frac{n!}{n!}, \\ x(n-1)! &= a_1(n-1)!+a_2\frac{(n-1)!}{2!}+\cdots+a_{n-1}\frac{(n-1)!}{(n-1)!}+\frac{a_n}{n}. \end{aligned}$$

So n is the smallest integer such that $n!x$ is an integer, i.e., $2013^{1000} \mid n!$, or equivalently $61^{1000} \mid n!$ because 61 is the largest prime divisor of 2013.

Since $\left\lfloor\frac{1000}{61}\right\rfloor = 16$, $n = (1000-16)\times 61 = 60024$.

34. Answer: 4.

Let a, b, c be fixed. Set $f(x) = ax^2+bx+c$. Then

$$f(-1) = a-b+c, \quad f(0) = c, \quad f(1) = a+b+c.$$

Solve the system to get

$$a = \frac{1}{2}f(-1) - f(0) + \frac{1}{2}f(1), \qquad b = -\frac{1}{2}f(-1)+\frac{1}{2}f(1).$$

Suppose $|f(x)| \leq 1$ for all $|x| \leq 1$. Then

$$\begin{aligned} |2ax+b| &= \left| \left(x - \frac{1}{2}\right) f(-1) - 2f(0)x + \left(x + \frac{1}{2}\right) f(1) \right| \\ &\leq \left| x - \frac{1}{2} \right| + 2|x| + \left| x + \frac{1}{2} \right| \\ &\leq \left| x - \frac{1}{2} \right| + \left| x + \frac{1}{2} \right| + 2 \leq 4. \end{aligned}$$

Moreover, $|2x^2 - 1| \leq 1$ whenever $|x| \leq 1$, and $|2x| = 4$ is achieved at $x = \pm 1$.

35. Answer: 81.

Let E_k denote the expected number of steps it takes to go from $k-1$ to k, $k = 2, \ldots, 100$. Then $E_{k+1} = \frac{1}{2}(1 + E_k + E_{k+1}) + \frac{1}{2}$, which implies $E_{k+1} = E_k + 2$.

It is clear that $E_2 = 1$. Then $E_3 = 3, E_4 = 5, \ldots, E_{10} = 17$. So

$$E = E_2 + E_3 + \cdots + E_{10} = 1 + 3 + \cdots + 17 = 81.$$