Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

Junior Section (First Round Solutions)

Multiple Choice Questions

1. Answer: (D)

   First note that $c = (8^2)^{27} = 8^{54}$, so we see that $c > a$. Next, $b = (4^2)^{41} = 4^{82}$ and $c = (4^3)^{27} = 4^{81}$. Therefore we have $b > c$. Consequently $b > c > a$.

2. Answer: (B)

$$\begin{aligned}\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c} &= \frac{a^2+b^2+c^2-bc-ca-ab}{abc}\\ &= \frac{(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca+a^2)}{2abc}\\ &= \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2abc}\\ &= \frac{1^2+1^2+2^2}{120}=\frac{1}{20}.\end{aligned}$$

3. Answer: (A)

   Note that $x^2+x+1=\dfrac{x^3-1}{x-1}$, so $x^2+x+1=0$ implies that $x^3=1$ and $x \neq 1$. Now

$$\begin{aligned}x^{49}+x^{50}+x^{51}+x^{52}+x^{53} &= x^{49}(1+x+x^2)+x^{51}(x+x^2)\\ &= x^{49}\times 0+(x^3)^{17}(-1)\\ &= 1^{17}\times(-1)=-1.\end{aligned}$$

4. Answer: (B)

   Let $\angle CAB = x$ and $\angle ABC = y$. Then $x+y = 180^\circ - 36^\circ = 144^\circ$.
   Now $\angle APB = 180^\circ - \frac{x+y}{2} = 108^\circ$.

5. Answer: (D)

   $xy-3x+5y=0$ is equivalent to $(x+5)(y-3)=-15$.
   If $x+5=a$ and $y-3=b$, then there are eight distinct pairs of integers a, b (counting signs) such that $ab=-15$.

6. Answer: (B)

Beatrice, being between Miss Poh and Miss Mak cannot be Miss Ong who was between Miss Lim and Miss Mak. This means that we have in order from the left, Miss Poh, Beatrice, Miss Mak, Miss Ong and Miss Lim. So Beatrice must be Miss Nai. Since Ellie was beside Miss Nai and also besides Miss Lim, she must be Miss Poh. This implies Cindy is Miss Lim and Amy was Miss Ong leaving Daisy as Miss Mak.

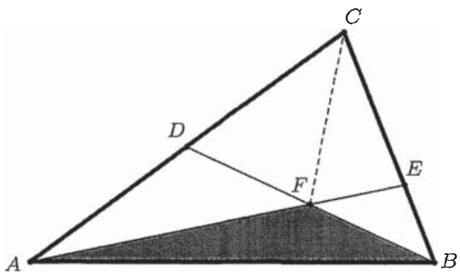
7. Answer: (D)

Construct a line joining C and F. Then using $[XYZ]$ to denote the area of $\triangle XYZ$, we know that $[ADF] = [DCF] = x$ and if $[BFE] = z$, then $[FCE] = 2z$.

Furthermore, we have $[ADB] = [DCB]$ i.e. $x + 1 = x + 3z$, so $z = \frac{1}{3}$.

Also, $2 \times [AEB] = [ACE]$ i.e. $2 + 2z = 2x + 2z$, so $x = 1$.

In conclusion, $[ABC] = 1 + 2x + 3z = 4$ units.



8. Answer: (C)

Join A to N. By symmetry, $AN + NM = MN + CN$, and the least value occurs when ANM is a straight line. Thus the least value is

$$\sqrt{AB^2 + BM^2} = \sqrt{8^2 + 6^2} = 10.$$

9. Answer: (D)

Since CE bisects $\angle BCD$, $\angle BCE = 45°$. Thus $\angle CEB = 45°$ also and $\triangle CBE$ is isosceles. Therefore $BC = BE$.

Now $\angle BCO = 45° + 15° = 60°$. As $CO = BO$, we conclude that $\triangle COB$ is equilateral. Thus $BC = BO = BE$ giving us an isosceles triangle OBE. Since $\angle OBE = 30°$, thus $\angle BOE = 75°$.

10. Answer: (D)

S being a multiple of 5 and 3 must end with '0' and has the sum of digits divisible by 3. Since $3 + 8 = 11$, the smallest positive k such that $k \times 11$ is divisible by 3 is 3. Thus $S = 300338880$ and the remainder is

$$0 - 8 + 8 - 8 + 3 - 3 + 0 - 0 + 3 = -5 \equiv 6 \pmod{11}.$$

Short Questions

11. Answer: 10000

$\sqrt{9999^2 + 19999} = \sqrt{9999^2 + 2 \times 9999 + 1} = \sqrt{(9999+1)^2} = 10000.$

12. Answer: 3

Let (α, β) be the point of intersection of the two graphs. Then

$$\beta = \alpha^2 + 2a\alpha + 6b = \alpha^2 + 2b\alpha + 6a.$$

It follows that $2(a-b)\alpha = 6(a-b)$. Since the two graphs intersect at only one point, we see that $a - b \neq 0$ (otherwise the two graphs coincide and would have infinitely many points of intersection). Consequently $2\alpha = 6$, and hence $\alpha = 3$.

13. Answer: 1822

The the number of multiples of 11 in the sequence $1, 2, ..., n$ is equal to $\lfloor \frac{n}{11} \rfloor$. Thus the answer to this question is $\left\lfloor \frac{20130}{11} \right\rfloor - \left\lfloor \frac{98}{11} \right\rfloor = 1830 - 8 = 1822.$

14. Answer: 108

Let $\angle ABC = \alpha$ and $\angle BAC = \beta$. Since $BA = BC$, we have $\angle BCA = \angle BAC = \beta$. As $EB = ED$, it follows that $\angle EDB = \angle EBD = \angle ABC = \alpha$. Then $\angle AFD = \angle ADF = \angle EDB = \alpha$ since $AD = AF$. Note that $\angle DAF = 180° - \beta$. In $\triangle ABC$, we have $\alpha + 2\beta = 180°$; and in $\triangle ADF$, we have $2\alpha + 180° - \beta = 180°$. From the two equations, we obtain $\alpha = 36°$. By considering $\triangle BDE$, we obtain $x = 180° - 2\alpha = 108°$.

15. Answer: 20

$$\begin{aligned}
&\frac{1}{a^2 - ac - ab + bc} + \frac{2}{b^2 - ab - bc + ac} + \frac{1}{c^2 - ac - bc + ab} \\
&= \frac{1}{(a-b)(a-c)} + \frac{2}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)} \\
&= \frac{c - b - 2(c-a) - (a-b)}{(a-b)(b-c)(c-a)} \\
&= \frac{-1}{(a-b)(b-c)} = 20.
\end{aligned}$$

16. Answer: 60

The equation $(x-2)^2 = 3(x+5)$ is equivalent to $x^2 - 7x - 11 = 0$. Thus $x_1 + x_2 = 7$ and $x_1x_2 = -11$. So

$$x_1x_2 + x_1^2 + x_2^2 = (x_1 + x_2)^2 - x_1x_2 = 7^2 - (-11) = 60.$$

17. Answer: 1041

Let the length of the side be s. Observe that since $6^2+8^2=10^2$ so $\angle AXY=90°$. This allows us to see that $\triangle ABX$ is similar to $\triangle XCY$. Thus $\frac{AX}{XY}=\frac{AB}{XC}$, i.e. $\frac{8}{6}=\frac{s}{s-BX}$. Solving this equation gives $s=4BX$ and we can then compute that

$$8^2=AB^2+BX^2=16BX^2+BX^2.$$

So $BX=\frac{8}{\sqrt{17}}$ and $s^2=16\times\frac{64}{17}=\frac{1024}{17}$. Thus $m+n=1041$.

18. Answer: 125

The inequality is equivalent to

$$(x-5)^2+(2x-y)^2\le 0.$$

Thus we must have $(x-5)=0$ and $(2x-y)=0$, hence $x^2+y^2=5^2+10^2=125$.

19. Answer: 6

Suppose Team B spent t minutes on the job. Then

$$\frac{t}{75}+\frac{90-t}{150}=1.$$

Thus $t=60$ minutes and so Team A completed $\frac{30}{150}=\frac{1}{5}$ of the job. So $m+n=6$.

20. Answer: 4

Taking reciprocals, we find that $\frac{1}{a}+\frac{1}{b}=3$, $\frac{1}{b}+\frac{1}{c}=4$ and $\frac{1}{a}+\frac{1}{c}=5$. Summing the three equations, we get

$$12=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=2\times\frac{ab+bc+ca}{abc}.$$

Hence $\frac{24abc}{ab+bc+ca}=4$.

21. Answer: 8052

$$(x_1+x_2)^2=(x_1-x_2)^2+4x_1x_2\ge 0+4\times 2013=8052.$$

If $x_1=x_2=\sqrt{2013}$, then $(x_1+x_2)^2=8052$.

22. Answer: 10

Let $x_1=\sqrt{45-\sqrt{2000}}$ and $x_2=\sqrt{45+\sqrt{2000}}$. Then $x_1^2+x_2^2=90$ and

$$x_1x_2=\sqrt{(45-\sqrt{2000})(45+\sqrt{2000})}=\sqrt{45^2-2000}=\sqrt{25}=5.$$

Thus

$$(x_1+x_2)^2=x_1^2+x_2^2+2x_1x_2=100.$$

As both x_1 and x_2 are positive, we have $x_1+x_2=10$.

23. Answer: 55

$(k-10)^{4026} = ((k-10)^2)^{2013} \geq 2013^{2013}$ is equivalent to to $(k-10)^2 \geq 2013$. As $k-10$ is an integer and $44^2 < 2013 < 45^2$, the minimum value of $k-10$ is 45, and thus the minimum value of k is 55.

24. Answer: 19

Rearranging the terms of the equation, we obtain

$$(1-4a)x = 2b+3.$$

Since the equation has more than one solution (i.e., infinitely many solutions), we must have $1-4a=0$ and $2b+3=0$. Therefore $a=\frac{1}{4}$ and $b=-\frac{3}{2}$. Consequently, $100a+4b=19$.

25. Answer: 18

First note that the product of any two different 2-digit numbers is greater than 100. Thus if a 2-digit number is chosen, then the two numbers adjacent to it in the circle must be single-digit numbers. Note that at most nine single-digit numbers can be chosen from S, and no matter how these nine numbers $1, 2, \ldots, 9$ are arranged in the circle, there is at most one 2-digit number in between them. Hence it follows that $n \leq 18$. Now the following arrangement

$$1, 49, 2, 33, 3, 24, 4, 19, 5, 16, 6, 14, 7, 12, 8, 11, 9, 10, 1$$

shows that $n \geq 18$. Consequently we conclude that the maximum value of n is 18.

26. Answer: 48

Let $x = \overline{abcd}$ and $y = \overline{dcba}$ where $a, d \neq 0$. Then

$$\begin{aligned} y - x &= 1000 \times d - d + 100 \times c - 10 \times c + 10 \times b - 100 \times b + a - 1000 \times a \\ &= 999(d-a) + 90(c-b) = 9\left(111(d-a) + 10(c-b)\right). \end{aligned}$$

So we have $111(d-a)+10(c-b) = 353$. Consider the remainder modulo 10, we obtain $d-a=3$, which implies that $c-b=2$. Thus the values of a and b determines the values of d and c respectively.

a can take on any value from 1 to 6, and b can take any value from 0 to 7, giving $6 \times 8 = 48$ choices.

27. Answer: 12

Let $2^8 + 2^{11} + 2^n = m^2$ and so

$$2^n = m^2 - 2^8(1+8) = (m-48)(m+48).$$

If we let $2^k = m+48$, then $2^{n-k} = m-48$ and we have

$$2^k - 2^{n-k} = 2^{n-k}(2^{2k-n}-1) = 96 = 2^5 \times 3.$$

This means that $n-k=5$ and $2k-n=2$, giving us $n=12$.

28. Answer: 208

Note that a positive integer k is a multiple of 4 if and only if the number formed by the last two digits of k (in the same order) is a multiple of 4. There are 12 possible multiples of 4 that can be formed from the digits $0,1,2,3,4,5,6$ without repetition, namely

$$20, 40, 60, 12, 32, 52, 04, 24, 64, 16, 36, 56.$$

If 0 appears in the last two digits, there are 5 choices for the first digit and 4 choices for the second digit. But if 0 does not appear, there are 4 choices for the first digit and also 4 choices for the second digit. Total number is

$$4 \times 5 \times 4 + 8 \times 4 \times 4 = 208.$$

29. Answer: 10571

$$\begin{aligned}\frac{m}{n} = \frac{1}{4}\sum_{k=5}^{1006}\frac{1}{k(k+1)} &= \frac{1}{4}\sum_{k=5}^{1006}\frac{1}{k} - \frac{1}{k+1} \\ &= \frac{1}{4}\left(\frac{1}{5} - \frac{1}{1007}\right) \\ &= \frac{501}{10070}.\end{aligned}$$

Since $\gcd(501, 10070) = 1$, we have $m + n = 10571$.

30. Answer: 3

Note that the units digit of $2013^1 + 2013^2 + 2013^3 + \cdots + 2013^{2013}$ is equal to the units digit of the following number

$$3^1 + 3^2 + 3^3 + \cdots + 3^{2013}.$$

Since $3^2 = 9, 3^3 = 27, 3^4 = 81$, the units digits of the sequence of $3^1, 3^2, 3^3, 3^4, \cdots, 3^{2013}$ are

$$\underbrace{3, 9, 7, 1, 3, 9, 7, 1, \cdots, 3, 9, 7, 1}_{2012 \text{ numbers}}, 3.$$

Furthermore the sum $3 + 9 + 7 + 1$ does not contribute to the units digit, so the answer is 3.

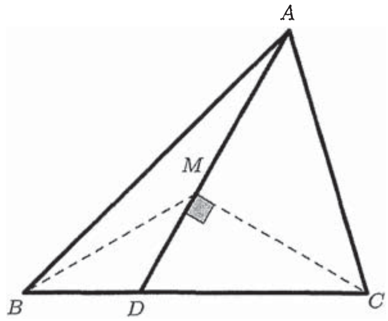
31. Answer: 75

Construct a point M on AD so that CM is perpendicular to AD. Join B and M.

Since $\angle ADC = 60°$, $\angle MCD = 30°$. As $\sin 30° = \frac{1}{2}$, so $2MD = DC$. This means that $BD = MD$ and $\triangle MDB$ is isosceles. It follows that $\angle MBD = 30°$ and $\angle ABM = 15°$.

We further observe that $\triangle MBC$ is also isosceles and thus $MB = MC$.

Now $\angle BAM = \angle BMD - \angle ABM = 15°$, giving us yet another isosceles triangle $\triangle BAM$. We now have $MC = MB = MA$, so $\triangle AMC$ is also isosceles. This allows us to calculate $\angle ACM = 45°$ and finally $\angle ACB = 30° + 45° = 75°$.



32. Answer: 221

We have $a^2+2ab-3b^2 = (a-b)(a+3b) = 41$. Since 41 is a prime number, and $a-b < a+3b$, we have $a - b = 1$ and $a + 3b = 41$. Solving the simultaneous equations gives $a = 11$ and $b = 10$. Hence $a^2 + b^2 = 221$.

33. Answer: 62

We first note that for $1 \leq r < k$, $\lfloor \frac{r}{k} \rfloor = 0$ and $\lfloor \frac{k}{k} \rfloor = 1$. The total number of terms up to $\lfloor \frac{N}{N} \rfloor$ is given by $\frac{1}{2}N(N+1)$, and we have the inequality

$$\frac{62(63)}{2} = 1953 < 2013 < 2016 = \frac{63(64)}{2}.$$

So the 2013$^{\text{th}}$ term is $\lfloor \frac{60}{63} \rfloor$, and the sum up to this term is just 62.

34. Answer: 1648

By the pigeonhole principle in any group of $365 \times 9 + 1 = 3286$ persons, there must be at least 10 persons who share the same birthday.

Hence solving $2n - 10 \geq 3286$ gives $n \geq 1648$. Thus the smallest possible n is 1648 since $2 \times 1647 - 10 = 3284 < 365 \times 9$, and it is possible for each of the 365 different birthdays to be shared by at most 9 persons.

35. Answer: 64

Let $d > 1$ be the highest common factor of $n - 11$ and $3n + 20$. Then $d \mid (n - 11)$ and $d \mid (3n + 20)$. Thus $d \mid [3n + 20 - 3(n - 11)]$, i.e., $d \mid 53$. Since 53 is a prime and $d > 1$, it follows that $d = 53$. Therefore $n - 11 = 53k$, where k is a positive integer, so $n = 53k + 11$. Note that for any k, $3n+20$ is a multiple of 53 since $3n+20 = 3(53k+11)+20 = 53(3k+1)$. Hence $n = 64$ (when $k = 1$) is the smallest positive integer such that $\text{HCF}(n-11, 3n+20) > 1$.