# Singapore Mathematical Society

## Singapore Mathematical Olympiad (SMO) 2007

## (Senior Section)

Tuesday, 29 May 2007 — 0930 – 1200 hrs

**Important:**

*Answer ALL 35 questions.*

*Enter your answers on the answer sheet provided.*

*For the multiple choice questions, enter your answers in the answer sheet by shading the bubbles containing the letters* (A, B, C, D *or* E) *corresponding to the correct answers.*

*For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.*

*No steps are needed to justify your answers.*

*Each question carries 1 mark.*

*No calculators are allowed.*

**PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO**

**Multiple Choice Questions**

1. Find the sum of the digits of the product $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\ldots\left(1+\frac{1}{2006}\right)\left(1+\frac{1}{2007}\right)$.

(A) 5
(B) 6
(C) 9
(D) 10
(E) 13

2. A bag contains $x$ green and $y$ red sweets. A sweet is selected at random from the bag and its colour noted. It is then replaced into the bag together with 10 additional sweets of the same colour. A second sweet is next randomly drawn. Find the probability that the second sweet is red.

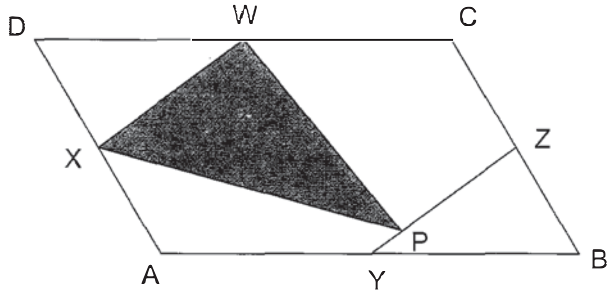
(A) $\frac{y+10}{x+y+10}$
(B) $\frac{y}{x+y+10}$
(C) $\frac{y}{x+y}$
(D) $\frac{x}{x+y}$
(E) $\frac{x+y}{x+y+10}$

3. What is the remainder when the number

$$(\underbrace{999\,999\,999\ldots999}_{2008\ 9\text{'s}})^{2007} - (\underbrace{333\,333\,333\ldots333}_{2008\ 3\text{'s}})^{2007}$$

is divided by 11?
(A) 0
(B) 2
(C) 4
(D) 6
(E) None of the above

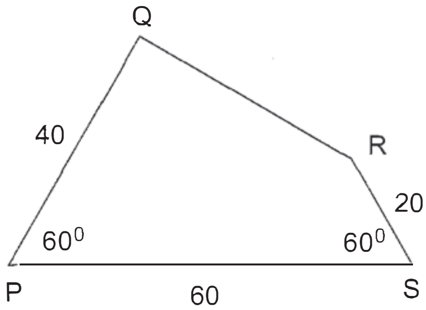
4. W, X, Y and Z are the midpoints of the four sides of parallelogram ABCD. P is a point on the line segment YZ. What percent of the area of parallelogram ABCD is triangle PXW?




(A) 50%
(B) 45%
(C) 30%
(D) 25%
(E) 20%

5. Four rods are connected together with flexible joints at their ends to make a quadrilateral as shown. Rods $PQ = 40$ cm, $RS = 20$ cm, $PS = 60$ cm and $\angle QPS = \angle RSP = 60^o$. Find $\angle QRS$.

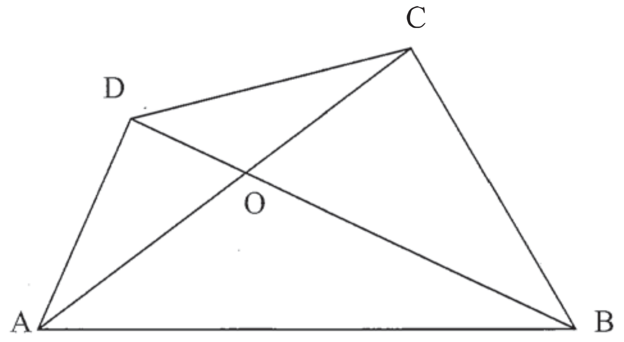
(A) $100^o$
(B) $105^o$
(C) $120^o$
(D) $135^o$
(E) $150^o$




6. When 2007 bars of soap are packed into $N$ boxes, where $N$ is a positive integer, there is a remainder of 5. How many possible values of $N$ are there?
(A) 14
(B) 16
(C) 18
(D) 20
(E) 13

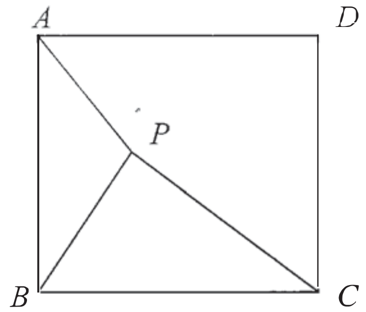
7. Suppose $a_n$ denotes the last two digits of $7^n$. For example, $a_2 = 49$, $a_3 = 43$. Find the value of $a_1 + a_2 + a_3 + \dots\dots\dots + a_{2007}$
(A) 50189
(B) 50199
(C) 50209
(D) 50219
(E) 50229

8. The diagram below shows a quadrilateral ABCD where AB = 10, BC = 6, CD = 8 and DA = 2. The diagonals AC and BD intersect at the point O and that $\angle COB = 45^0$. Find the area of the quadrilateral ABCD.




(A) 28
(B) 29
(C) 30
(D) 31
(E) 32

9. In the following diagram, $ABCD$ is a square with $PA = a$, $PB = 2a$ and $PC = 3a$. Find $\angle APB$.




(A) $120^0$
(B) $130^0$
(C) $135^0$
(D) $140^0$
(E) $145^0$

10. What is the largest possible prime value of $n^2 - 12n + 27$, where $n$ ranges over all positive integers?
(A) 91
(B) 37
(C) 23
(D) 17
(E) 7

## Short Questions

11. Suppose that $\log_2[\log_3(\log_4 a)] = \log_3 [\log_4 (\log_2 b)] = \log_4 [\log_2(\log_3 c)] = 0$. Find the value of $a + b + c$.

12. Find the unit digit of $17^{17} \text{ x } 19^{19} \text{ x } 23^{23}$.

13. Given that $x + y = 12$ and $xy = 50$, find the exact value of $x^2 + y^2$.

14. Suppose that $(21.4)^a = (0.00214)^b = 100$. Find the value of $\frac{1}{a} - \frac{1}{b}$.

15. Find the value of $100(\sin 253^\circ \sin 313^\circ + \sin 163^\circ \sin 223^\circ)$

16. The letters of the word MATHEMATICS are rearranged in such a way that the first four letters of the arrangement are all vowels. Find the total number of distinct arrangements that can be formed in this way.
(Note: The vowels of English language are A, E, I, O, U)

17. Given a set S = {1, 2, 3, …, 199, 200 }. The subset A = $\{a, b, c\}$ of S is said to be "*nice*" if $a + c = 2b$. How many "*nice*" subsets does S have?
(Note: The order of the elements inside the set does not matter. For example, we consider $\{a, b, c\}$ or $\{a, c, b\}$ or $\{c, b, a\}$ to be the same set.)

18. Find the remainder when $2^{55} + 1$ is divided by 33.

19. Given that the difference between two 2-digit numbers is 58 and these last two digits of the squares of these two numbers are the same, find the smaller number.

20. Evaluate $256 \sin 10^0 \sin 30^0 \sin 50^0 \sin 70^0$.

21. Find the greatest integer less than or equal to $\left(2+\sqrt{3}\right)^3$.

22. Suppose that $x_1$, $x_2$ and $x_3$ are the three roots of $(11-x)^3+(13-x)^3=(24-2x)^3$. Find the value of $x_1+x_2+x_3$.

23. In $\Delta$ABC, $\angle$CAB = 30° and $\angle$ABC = 80°. The point M lies inside the triangle such that $\angle$MAC = 10° and $\angle$MCA = 30°. Find $\angle$BMC in degrees.

24. How many positive integer $n$ less than 2007 can we find such that $\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\left[\frac{n}{6}\right]=n$ where $[x]$ is the greatest integer less than or equal to $x$?
(For example, [2.5] = 2; [5] = 5; [-2.5] = -3 etc.)

25. In $\Delta ABC$, let $AB=c$, $BC=a$ and $AC=b$. Suppose that $\frac{b}{c-a}-\frac{a}{b+c}=1$, find the value of the greatest angle of $\Delta ABC$ in degrees.

26. Find the number of integers $N$ satisfying the following two conditions:
    (i) $1 \le N \le 2007$; and
    (ii) either $N$ is divisible by 10 or 12 (or both).

27. Suppose $a$ and $b$ are the roots of $x^2 + x\sin\alpha + 1 = 0$ while $c$ and $d$ are the roots of the equation $x^2 + x\cos\alpha - 1 = 0$. Find the value of $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}$.

28. A sequence $\{a_n\}$ is defined by $a_1 = 2$, $a_n=\frac{1+a_{n-1}}{1-a_{n-1}}$, $n \ge 2$. Find the value of $-2008\,a_{2007}$.

29. Let $x$, $y$ and $z$ be three real numbers such that $xy+yz+xz=4$. Find the least possible value of $x^2+y^2+z^2$.

30. P is the set {1, 2, 3, …, 14, 15 }. If A = $\{a_1, a_2, a_3\}$ is a subset of P where $a_1 < a_2 < a_3$ such that $a_1 + 6 \le a_2 + 3 \le a_3$. How many such subsets are there of P ?

31. It is given that $x$ and $y$ are two real numbers such that

$$(x+y)^4+(x-y)^4=4112 \text{ and } x^2-y^2=16.$$

Find the value of $x^2+y^2$.

32. Let $A$ be an angle such that $\tan 8A=\dfrac{\cos A-\sin A}{\cos A+\sin A}$. Suppose $A=x^{\circ}$ for some positive real number $x$. Find the smallest possible value of $x$.

33. Find the minimum value of $\sum_{k=1}^{100}|n-k|$, where $n$ ranges over all positive integers.

34. Find the number of pairs of positive integers $(x, y)$ are there which satisfy the equation $2x + 3y = 2007$.

35. If $S=\dfrac{1}{1+1^2+1^4}+\dfrac{2}{1+2^2+2^4}+\dfrac{3}{1+3^2+3^4}+\cdots\cdots\cdots+\dfrac{200}{1+200^2+200^4}$, find the value of $80402\times S$.