Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Junior Section)

Tuesday, 29 May 2007 0930-1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter only the letters (A, B, C, D, or E) corresponding to the correct answers in the answer sheet.

For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

1. Among the four statements on integers below,

"If $a < b$ then $a^2 < b^2$"; "$a^2 > 0$ is always true";

"$-a < 0$ is always true"; "If $ac^2 < bc^2$ then $a < b$",

how many of them are correct?

(A) 0; (B) 1; (C) 2; (D) 3; (E) 4.

2. Which of the following numbers is odd for any integer values of k?

(A) $2007 + k^3$; (B) $2007 + 7k$; (C) $2007 + 2k^2$; (D) $2007 + 2007k$; (E) $2007k$.

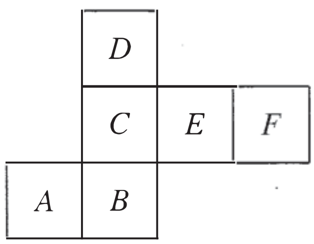
3. In a school, all 300 Secondary 3 students study either Geography, Biology or both Geography and Biology. If 80% study Geography and 50% study Biology, how many students study both Geography and biology?

(A) 30; (B) 60; (C) 80; (D) 90; (E) 150.

4. An unbiased six-sided dice is numbered 1 to 6. The dice is thrown twice and the two scores added. Which of the following events has the highest probability of occurrence?

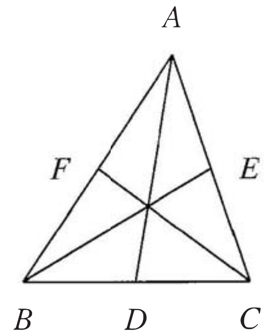
(A) The total score is a prime number; (B) The total score is a multiple of 4;
(C) The total score is a perfect square; (D) The total score is 7;
(E) The total score is a factor of 12.

5. The cardboard below can be cut out and folded to make a cube. Which face will then be opposite the face marked A?



(A) B; (B) C; (C) D; (D) E; (E) F.

6. How many triangles can you find in the following figure?

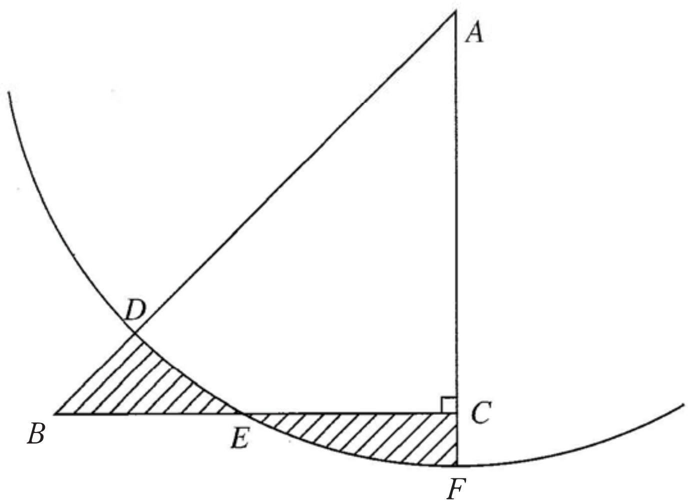


(A) 7; (B) 10; (C) 12; (D) 16; (E) 20.

7. Suppose x_1, x_2 and x_3 are roots of $(11-x)^3+(13-x)^3=(24-2x)^3$. What is the sum of $x_1+x_2+x_3$?

(A) 30; (B) 36; (C) 40; (D) 42; (E) 44.

8. In the following right-angled triangle ABC, $AC=BC=1$ and DEF is an arc of a circle with center A. Suppose the shaded areas BDE and CEF are equal and $AD=\frac{x}{\sqrt{\pi}}$. What is the value of x?



(A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

9. Suppose

$$\frac{1}{x} = \frac{2}{y+z} = \frac{3}{z+x} = \frac{x^2 - y - z}{x+y+z}.$$

What is the value of $\frac{z-y}{x}$?

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

10. Suppose $x^2 - 13x + 1 = 0$. What is the last digit of $x^4 + x^{-4}$?

(A) 1; (B) 3; (C) 5; (D) 7; (E) 9.

11. In a triangle ABC, it is given that $AB = 1$ cm, $BC = 2007$ cm and $AC = a$ cm, where a is an integer. Determine the value of a.

12. Find the value (in the simplest form) of $\sqrt{21 + 12\sqrt{3}} - \sqrt{21 - 12\sqrt{3}}$.

13. Find the value of

$$\frac{2007^2 + 2008^2 - 1993^2 - 1992^2}{4}.$$

14. Find the greatest integer N such that

$$N \leq \sqrt{2007^2 - 20070 + 31}.$$

15. Suppose that x and y are non-zero real numbers such that

$$\frac{x}{3} = y^2 \quad \text{and} \quad \frac{x}{9} = 9y.$$

Find the value of $x + y$.

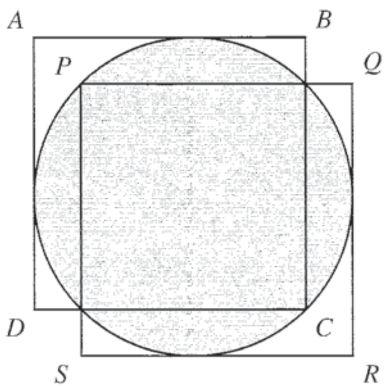
16. Evaluate the sum

$$\frac{2007}{1 \times 2} + \frac{2007}{2 \times 3} + \cdots + \frac{2007}{2006 \times 2007}.$$

17. Find the sum of the digits of the product

$$(\underbrace{111111111\ldots111}_{2007\ 1's}) \times 2007$$

18. The diagram shows two identical squares, $ABCD$ and $PQRS$, overlapping each other in such a way that their edges are parallel, and a circle of radius $(2 - \sqrt{2})$ cm covered within these squares. Find the length of the square $ABCD$.

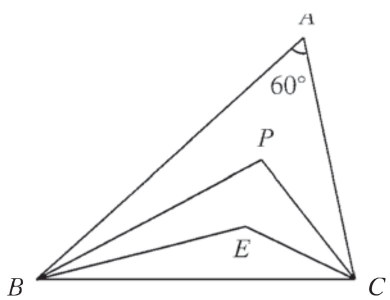


19. When 2007 bars of soap are packed into N boxes of equal size, where N is an integer strictly between 200 and 300, there are extra 5 bars remaining. Find N.

20. Suppose that $a + x^2 = 2006$, $b + x^2 = 2007$ and $c + x^2 = 2008$ and $abc = 3$. Find the value of

$$\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}.$$

21. The diagram below shows a triangle ABC in which $\angle A = 60°$, BP and BE trisect $\angle ABC$; and CP and CE trisect $\angle ACB$. Let the angle $\angle BPE$ be $x°$. Find x.



22. Suppose that $x - y = 1$. Find the value of $x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4$.

23. How many ordered pairs of integers (m, n) where $0 < m < n < 2008$ satisfy the equation $2008^2 + m^2 = 2007^2 + n^2$?

24. If $x + \sqrt{xy} + y = 9$ and $x^2 + xy + y^2 = 27$, find the value of $x - \sqrt{xy} + y$.

25. Appending three digits at the end of 2007, one obtains an integer N of seven digits. In order to get N to be the minimal number which is divisible by 3, 5 and 7 simultaneously, what are the three digits that one would append?

26. Find the largest integer n such that $n^{6021} < 2007^{2007}$.

27. Find the value of

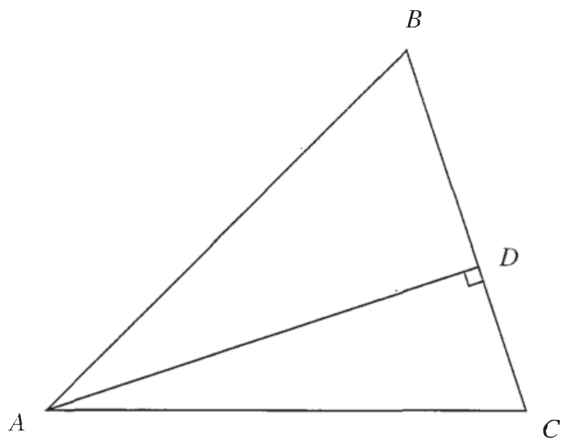
$$\frac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15}$$

when $x = \sqrt{19 - 8\sqrt{3}}$.

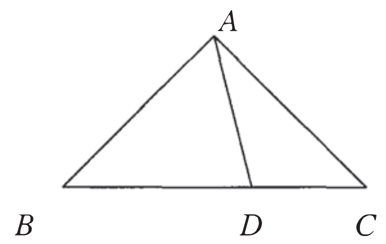
28. Find the value of a such that the two equations $x^2 + ax + 1 = 0$ and $x^2 - x - a = 0$ have one common real root.

29. Odd integers starting from 1 are grouped as follows: $(1), (3, 5), (7, 9, 11), (13, 15, 17, 19), \ldots,$ where the n-th group consists of n odd integers. How many odd integers are in the same group which 2007 belongs to?

30. In $\triangle ABC$ $\angle BAC = 45^\circ$. D is a point on BC such that AD is perpendicular to BC. If $BD = 3$ cm and $DC = 2$ cm, and the area of the $\triangle ABC$ is x cm^2. Find the value of x.

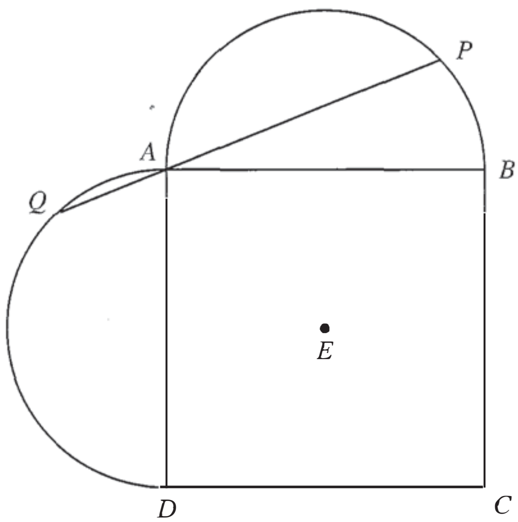


31. In $\triangle ABC$ (see below), $AB = AC = \sqrt{3}$ and D is a point on BC such that $AD = 1$. Find the value of $BD \cdot DC$.



32. Find the last digit of $2^{2^{2007}} + 1$.

33. In the following diagram, $ABCD$ is a square, and E is the center of the square $ABCD$. P is a point on a semi-circle with diameter AB. Q is a point on a semi-circle with diameter AD. Moreover, Q, A and P are collinear (that is, they are on the same line). Suppose $QA = 14$ cm, $AP = 46$ cm, and $AE = x$ cm. Find thè value of x.



34. Find the smallest positive integer n such that $n(n + 1)(n + 2)$ is divisible by 247.

35. Find the largest integer N such that both $N + 496$ and $N + 224$ are perfect squares.