Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 2)

Saturday, 30 June 2007 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let $a_1, a_2, \ldots, a_n$ be n real numbers whose squares sum to 1. Prove that for any integer $k \geq 2$, there exists n integers $x_1, x_2, \ldots, x_n$, each with absolute value $\leq k-1$ and not all 0, such that
$$\left|\sum_{i=1}^{n} a_i x_i\right| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

2. If $a_1,\ a_2,\ \ldots a_n$ are distinct integers, prove that $(x-a_1)(x-a_2)\ldots(x-a_n)-1$ cannot be expressed as a product of two polynomials, each with integer coefficients and of degree at least 1.

3. Let A_1, B_1 be two points on the base AB of an isosceles triangle ABC, with $\angle C > 60^\circ$, such that $\angle A_1CB_1 = \angle ABC$. A circle externally tangent to the circumcircle of $\triangle A_1B_1C$ is tangent to the rays CA and CB at points A_2 and B_2, respectively. Prove that $A_2B_2 = 2AB$.

4. Let $\mathbb{N}$ be the set of positive integers, i.e., $\mathbb{N} = \{1, 2, \ldots\}$. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$f\big(f(m) + f(n)\big) = m + n \quad \text{for all } m, n \in \mathbb{N}.$$

5. Find the largest positive integer x such that x is divisible by all the positive integers $\leq \sqrt[3]{x}$.