Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Senior Section Solution)

1. (A)

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)..\left(1+\frac{1}{2006}\right)\left(1+\frac{1}{2007}\right)=\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\cdots\frac{2008}{2007}=\frac{2008}{2}=1004,$$

so the sum of the digits is 5.

2. (C)

Probability $=\left(\frac{x}{x+y}\times\frac{y}{x+y+10}\right)+\left(\frac{y}{x+y}\times\frac{y+10}{x+y+10}\right)$, which upon simplification yields (C) as the answer.

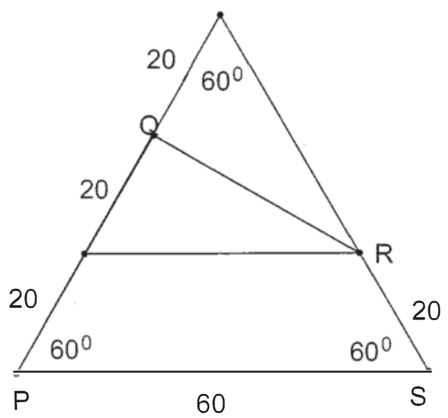
3. (A)

Observe that each of the two numbers is divisible by 11, hence the difference is also divisible by 11. Hence the remainder is zero.

4. (D)

Parallelogram WXYZ is half of ABCD. Triangle PXW is half of WXYZ. Thus, PXW is a quarter of ABCD. Hence (D)

5. (E)



We extend the given figure to the above, into an equilateral triangle. It is not possible to show that $\angle PQR = 90^{\circ}$. Hence, $\angle QRS = 30^{\circ} + 120^{\circ}$.

6. (A)

$2007 - 5 = 2002$. N is a factor of 2002 and $2002 = 2 \times 7 \times 11 \times 13$.

There are altogether 2x2x2x2 = 16 factors of N. However, N must exceed 5. So, N cannot be 1 or 2. Hence there are 14 possible choices of N.

7. (B)

Observe that a_n repeats itself as shown in the following table.

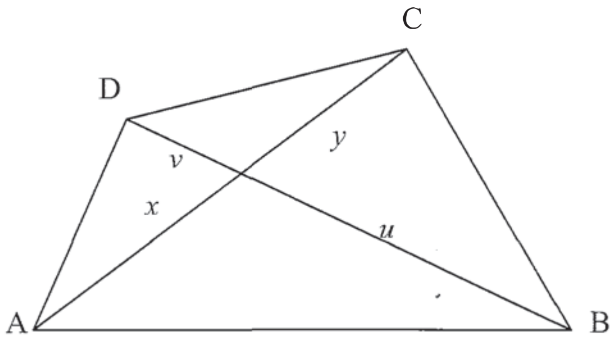
n	1	2	3	4	5	6	7	....
a_n	07	49	43	01	07	49	43	....

Further, $a_1 + a_2 + a_3 + a_4 = 100$.

Notice that 2007 = 4 x 501 + 3.

Hence the sum equals 501 x 100 + 7 + 49 + 43 = 50199.

8. (D)



Area of quadrilateral = $\frac{1}{2}(xu + yu + yv + xv)\sin 45^0$.

Using Cosine Rule, we have

$$x^2 + u^2 + 2xu\cos 45^0 = 10^2 \quad (1)$$

$$u^2 + y^2 - 2uy\cos 45^0 = 6^2 \quad (2)$$

$$y^2 + v^2 + 2yv\cos 45^0 = 8^2 \quad (3)$$

$$x^2 + v^2 - 2xv\cos 45^0 = 2^2 \quad (4)$$

(1) – (2) + (3) – (4):

$$2(xu + yu + yv + vx)\cos 45^0 = 10^2 - 6^2 + 8^2 - 2^2$$

Hence area of quadrilateral = $\frac{100 - 36 + 64 - 4}{4} = 31$.

9. (C)

Let $AB = x$ and $\angle PAB = \alpha$. By using Cosine Rule and Pythagoras' Theorem we have

$$x^2 + a^2 - 2ax\cos\alpha = (2a)^2$$

$$(x - a\cos\alpha)^2 + (x - a\sin\alpha)^2 = (3a)^2.$$

We obtain from the above two equations that

$$\cos\alpha = \frac{x^2 - 3a^2}{2ax} \quad \text{and} \quad \sin\alpha = \frac{x^2 - 5a^2}{2ax}.$$

Using $\sin^2\alpha + \cos^2\alpha = 1$, we have

$$\left(x^2 - 3a^2\right)^2 + \left(x^2 - 5a^2\right)^2 = 4a^2x^2$$

Therefore $x^4 - 10a^2x^2 + 17a^4 = 0$.

Solving, we have $x^2 = \left(5 \pm 2\sqrt{2}\right)a^2$, but since $x > a$, we have $x^2 = \left(5 + 2\sqrt{2}\right)a^2$.

Hence $\cos \angle APB = -\frac{\sqrt{2}}{2}$, so the required angle is 135^0.

10. (E)
Note that $n^2 - 12n + 27 = (n - 9)(n - 3)$. For this number to be a prime, either $n = 10$, in which case $n^2 - 12n + 27 = 7$; or $n = 2$, in which case $n^2 - 12n + 27 = 7$. Since a prime number cannot be factorized in other ways, we know that 7 is the only answer.

11. Answer: 89
$\log_2 [\log_3 (\log_4 a)] = 0$ implies $\log_3 (\log_4 a) = 1$. Hence $\log_4 a = 3$, hence $a = 64$. Similarly, $b = 16$ and $c = 9$. Therefore $a + b + c = 89$.

12. Answer: 1
$17^{17} \equiv 7^{17} \equiv 7 \pmod{10}$
$19^{19} \equiv 9^{19} \equiv 9 \pmod{10}$
$23^{23} \equiv 3^{23} \equiv 7 \pmod{10}$
Since $7 \times 9 \times 7 = 441 \equiv 1 \pmod{10}$, the unit digit is 1.

13. Answer: 44
Use the identity $x^2 + y^2 = (x + y)^2 - 2xy = 144 - 100 = 44$.
(Note: In this case, one may not be able to obtain the answer by guess-and-check; the values of x and y are not even real numbers)

14. Answer: 2
We have $a = \frac{2}{\log_{10} 21.4}$ and $b = \frac{2}{(\log_{10} 21.4) - 4}$. Hence direct computation yields $\frac{1}{a} - \frac{1}{b} = 2$.

15. Answer: 50
$$\begin{aligned}
&100(\sin 253^\circ \sin 313^\circ + \sin 163^\circ \sin 223^\circ) \\
&\quad = 100(\sin 73^\circ \sin 47^\circ - \sin 17^\circ \sin 43^\circ) \\
&\quad = 100(\sin 47^\circ \cos 17^\circ - \cos 47^\circ \sin 17^\circ) \\
&\quad = 100 \sin(47^\circ - 17^\circ) = 100 \sin 30^\circ \\
&\quad = 50
\end{aligned}$$

16. Answer: 15 120

Total number of rearrangements = $\frac{4!}{2!} \times \frac{7!}{2!2!} = 12 \times 1260 = 15120.$

17. Answer: 9900

Since $a + c$ is even, a and c have the same parity (either both odd or both even). There are 100 odd and 100 even numbers in S. Once a and c are chosen, b is determined. The number of "*nice*" subsets is $2\binom{100}{2} = 100(99) = 9900$

18: Answer: 0

Note that $x^{11} + 1$ is divisible by $x + 1$ by Factor Theorem. Hence $2^{55} + 1 = 32^{11} + 1$ is divisible by $32 + 1 = 33$, i.e. $2^{55} + 1$ is divisible by 33.

19. Answer: 21

Let the numbers be n and m where $n > m$.

$n - m = 58$

$n^2 - m^2 = (n - m)(n + m) = 58(n + m)$ is a multiple of 100, that is,

$58(n + m) = 100k$ for some integer k. This simplifies to

$29(n + m) = 50k$

Since gcd(29, 50) = 1, we must have $n + m = 50p$ for some positive integer p. Solving the above simultaneously with $n - m = 58$ and bearing in mind that both m and n are two digit numbers, only $n = 79$ and $m = 21$ satisfy the question. Thus the smaller number is 21.

20. Answer: 16

$256 \sin 10^0 \sin 30^0 \sin 50^0 \sin 70^0$

$$= \frac{2 \times 256 \sin 10^0 \cos 10^0 \sin 30^0 \cos 40^0 \cos 20^0}{2\cos 10^0}$$

$$= \frac{128 \sin 20^0 \cos 20^0 \cos 40^0}{2\cos 10^0}$$

$$= \frac{64 \sin 40^0 \cos 40^0}{2\cos 10^0}$$

$$= \frac{32 \sin 80^0}{2\cos 10^0}$$

$= 16.$

21. Answer: 51

Using binomial theorem it is easy to see that

$$\left(2+\sqrt{3}\right)^3 + \left(2-\sqrt{3}\right)^3 = 52,$$

and that $0 < 2-\sqrt{3} < 1$, or $0 < \left(2-\sqrt{3}\right)^3 < 1$ so that we have

$51 < \left(2+\sqrt{3}\right)^3 < 52.$

22. Answer: 36

Let $a = 11 - x$ and $b = 13 - x$. Hence the equation becomes

$$a^3 + b^3 = (a + b)^3 \qquad (1)$$

Using the binomial expansion

$$(a + b)^3 = a^3 + b^3 + 3ab\,(a + b),$$

Equation (1) becomes

$$ab\,(a + b) = 0 \qquad (2)$$

Thus the solution of (1) becomes

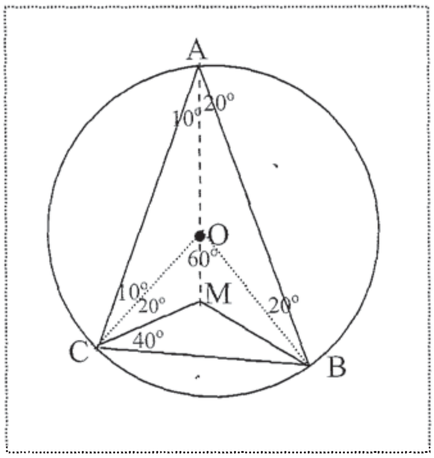
$a = 0$, $b = 0$ or $a + b = 0$

or equivalently, the solution of the original equation becomes $x = 11$, 12 or 13.

Hence $x_1 + x_2 + x_3 = 11 + 12 + 13 = 36$.

23. Answer: 110^0.

Construct a circumcircle of the triangle ABC, with O as the centre as shown.



Note that $\angle ACB = 70^o$

Since $OC = OB$ and $\angle COB = 60^o \Rightarrow \angle OCB = 60^o$

ΔCOB is an equilateral triangle.

Thus $\angle OCA = 70^o - 60^o = 10^o = \angle OAC$

But $\angle MAC = 10^o$ (given). So AOM lies on a straight line.

$\angle AOC = 160^o \Rightarrow \angle COM = 20^o$.

Since $\angle MCA = 30^o$ (given) and $\angle OCA = 10^o$, thus $\angle MCO = 20^o$. That means ΔMCO is an isosceles triangle and $\angle MCB = 70 - 30 = 40^o$

So BM is the perpendicular bisector of the equilateral triangle OBC. Thus

$\angle OBM = 60^o \div 2 = 30^o$

$\angle BMC = 180^o - 40^o - 30^o = 110^o$

24. Answer: 334

$$\left[\frac{n}{2}\right] \le \frac{n}{2}, \ \left[\frac{n}{3}\right] \le \frac{n}{3}, \ \left[\frac{n}{6}\right] \le \frac{n}{6}$$

Given $\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\left[\frac{n}{6}\right]=n$ but we have $\frac{n}{2}+\frac{n}{3}+\frac{n}{6}=n$

So $\left[\frac{n}{2}\right]=\frac{n}{2}$, $\left[\frac{n}{3}\right]=\frac{n}{3}$, $\left[\frac{n}{6}\right]=\frac{n}{6}$

Thus n must be a multiple of 6

There are $\left[\frac{2007}{6}\right]=334$ of them.

25. Answer: 120^0.

It is given that $\frac{b}{c-a}-\frac{a}{b+c}=1$. This can be rearranged into $b^2+a^2-c^2=-ab$.

By using cosine rule for triangle, $\cos C=\frac{b^2+a^2-c^2}{2ab}=-\frac{1}{2}$. Hence $C=120^0$ and must be the greatest angle.

26. Answer: 334

Number of integers divisible by 10 = $\left[\frac{2007}{10}\right]=200$.

Number of integers divisible by 12 = $\left[\frac{2007}{12}\right]=167$

Number of integers divisible by both 12 and 10 = $\left[\frac{2007}{60}\right]=33$.

By the principle of inclusion and exclusion, the number of integers divisible by either 10 or 12 (or both) = 200 + 167 – 33 = 334.

27. Answer: 1

We have $ab=1$. Hence $a^2=\frac{1}{b^2}$ and $b^2=\frac{1}{a^2}$.

Also, $cd=-1$. Hence $d^2=\frac{1}{c^2}$ and $c^2=\frac{1}{d^2}$.

Hence
$$\begin{aligned}\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2} &= a^2+b^2+c^2+d^2\\ &= (a+b)^2-2ab+(c+d)^2-2cd\\ &= \sin^2\alpha-2+\cos^2\alpha+2\\ &= 1\end{aligned}$$

28. Answer: 1004

Since $a_1 = 2$, $a_2 = -3$, $a_3 = -\frac{1}{2}$, $a_4 = \frac{1}{3}$, $a_5 = 2$, thus we consider

$$a_{n+4} = \frac{1+a_{n+3}}{1-a_{n+3}} = \frac{1+\frac{1+a_{n+2}}{1-a_{n+2}}}{1-\frac{1+a_{n+2}}{1-a_{n+2}}} = -\frac{1}{a_{n+2}} = -\left(\frac{1-a_{n+1}}{1+a_{n+1}}\right)$$

$$= -\frac{1-\frac{1+a_n}{1-a_n}}{1+\frac{1+a_n}{1-a_n}} = a_n$$

$a_{2007} = a_{4\text{x}501+3} = a_3 = -\frac{1}{2}$

$-2008\, a_{2007} = 1004$

29. Answer: 4

Use the identity

$x^2 + y^2 + z^2 - (xy + yz + xz) = \frac{1}{2}\left((x-y)^2 + (y-z)^2 + (x-z)^2\right) \geq 0$, the answer follows immediately.

30. Answer: 165

Supposing there are x_1 numbers smaller than a_1, x_2 numbers between a_1 and a_2 , x_3 numbers between a_2 and a_3 and x_4 numbers greater than a_3 .

Finding the number of possible subsets of A is equivalent to finding the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 12$ with the conditions that $x_1 \geq 0$, $x_2 \geq 2$, $x_3 \geq 2$ and $x_4 \geq 0$.

The number of solution of this latter equation is equivalent to the number of solutions of the equation $y_1 + y_2 + y_3 + y_4 = 8$, where y_1, y_2, y_3 and y_4 are nonnegative integers.

Hence the answer is $\binom{11}{3} = 165$.

31. Answer: 34

By binomial theorem, one sees that

$$(x+y)^4 + (x-y)^4 = 2(x^4 + 6x^2y^2 + y^4)$$

Thus the first equation becomes

$$2(x^4 + 6x^2y^2 + y^4) = 4112$$

$$x^4 + 6x^2y^2 + y^4 = 2056$$

$$(x^2 - y^2)^2 + 8x^2y^2 = 2056$$

$$(16)^2 + 8x^2y^2 = 2056$$

$$x^2y^2 = 2256$$

Therefore,

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 1156$$

Hence $x^2 + y^2 = \sqrt{1156} = 34$.

32. Answer: 5

Since $\frac{\sin 8A}{\cos 8A} = \tan 8A = \frac{\cos A - \sin A}{\cos A + \sin A}$, by rearranging we obtain

$$\sin 8A\,(\cos A + \sin A) = \cos 8A\,(\cos A - \sin A)$$

$$\sin 8A \cos A + \cos 8A \sin A = \cos 8A \cos A - \sin 8A \sin A$$

$$\sin (8A + A) = \cos (8A + A)$$

$$\sin 9A = \cos 9A$$

which reduces to

$$\tan 9A = 1$$

The smallest possible of $9A = 45^0$, which gives $x = 5$.

33. Answer: 2500

It is given that n is a positive integer.

For $n \ge 100$, $\sum_{k=1}^{100} |n - k| = 100n - 5050$, so that its minimum value is 4950 which occurs at $n = 100$.

For $n < 100$, $\sum_{k=1}^{100} |n - k| = n^2 - 101n + 5050$. If n is a positive integer, its minimum value can occur at either $n = 50$ or $n = 51$ only. By direct checking, its minimum value is 2500.

34. Answer: 334

$2x + 3y = 2007 \Rightarrow 2x = 2007 - 3y = 3(669 - y)$.

Since 2 and 3 are relatively prime, it follows that x is divisible by 3. Write $x = 3t$, where t is a positive integer.

The equation reduces to $y = 669 - 2t$.

Since $669 - 2t > 0$, we have $t < 334.5$, and since t is a positive integer, we have $1 \le t \le 334$.

Conversely, for any positive integer t satisfying $1 \le t \le 334$, it is easily seen that $(3t, 669 - 2t)$ is a pair of positive integers which satisfy the given equation.

Therefore there are 334 pairs of positive integers satisfying the given equations.

35. Answer: 40 200

Note that $\dfrac{k}{1+k^2+k^4} = \dfrac{1}{2}\left[\dfrac{1}{k(k-1)+1} - \dfrac{1}{k(k+1)+1}\right]$ for all integers k.

Hence the required sum can be found by the method of difference

$$= \frac{1}{2}\left(\frac{1}{1\times 0+1} - \frac{1}{1\times 2+1} + \frac{1}{2\times 1+1} - \frac{1}{2\times 3+1} + \frac{1}{3\times 2+1} - \frac{1}{3\times 4+1} + \ldots\ldots + \ldots\ldots + \frac{1}{200\times 199+1} - \frac{1}{200\times 201+1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{40201}\right)$$

$$= \frac{20100}{40201}$$

Hence 80402 x S = 40 200.