Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Junior Section Solutions)

1. Ans: (B)

   Only the last statement is correct: $ac^2 < bc^2$ implies $c^2 > 0$, hence $a < c$. For other statements, counterexamples can be take as $a = -2, b = -1$; $a = 0$ and $a = 0$ respectively.

2. Ans: (C)

   Because $2k^2$ is always even, thus $2007 + 2k^2$ is always odd. For other statements, either $k = 1$ or $k = 0$ gives a counterexample.

3. Ans: (D)

   Use Inclusion and Exclusion Principle.

4. Ans: (A)

   The number of occurrences for each event is: $15, 9, 7, 6$ and 12 respectively.

5. Ans: (D)

   Just imagine.

6. Ans: (D)

   Just count: Label the "center" O. There are 6 triangles like $\triangle AFO$; 3 like $\triangle AOB$; 6 like $\triangle ABD$ and 1 like $\triangle ABC$. Total: 16.

7. Ans: (B)

   Let $a = 11 - x$ and $b = 13 - x$. We have $a^3 + b^3 = (a+b)^3$. Simplify: $3ab(a+b) = 0$. Replacing a, b back in terms of x, we found the three roots are $11, 12$ and 13. Thus $x_1 + x_2 + x_3 = 36$.

8. Ans: (B)

   Since the area of sector ADF and $\triangle ABC$ are equal, we have

   $$\frac{1}{2}\left(\frac{x}{\sqrt{\pi}}\right)^2\frac{\pi}{4} = \frac{1}{2}.$$

   The result follows.

9. Ans: (B)

$$\frac{1}{x} = \frac{2}{y+z} \text{ and } \frac{1}{x} = \frac{3}{z+x}$$

tells us

$$\frac{y}{x} + \frac{z}{x} = 2 \text{ and } \frac{z}{x} + 1 = 3.$$

Thus $y = 0$ and $\frac{z}{x} = 2$. Note we didn't use the last equality, but $x = -1; y = 0$ and $z = -2$ satisfy all conditions.

10. Ans: (D)

By assumption $x + \frac{1}{x} = 13$. Thus $x^2 + \frac{1}{x^2} = 13^2 - 2 = 167$. Similarly, $x^4 + \frac{1}{x^4} = 167^2 - 2$, whose last digit is 7.

11. Ans: 2007.

Use $a < 2007 + 1$ and $2007 < a + 1$.

12. Ans: 6.

Use $21 \pm 12\sqrt{3} = (\sqrt{12} \pm 3)^2$.

13. Ans: 30000.

Using $a^2 - b^2 = (a+b)(a-b)$, $2008^2 - 1993^2 = 4001 \times 15$ and $2007^2 - 1992^2 = 3999 \times 15$. The result follows.

14. Ans: 2002.

By completion of square, $2007^2 - 20070 + 31 = (2007 - 5)^2 + 6$. The result follows.

15. Ans: 2214.

Eliminating x, we get $3y^2 = 81y$. Then $y = 27$ since $y \neq 0$. Thus $x = 2187$. The result follows.

16. Ans: 2006.

Using $\frac{1}{k \cdot (k+1)} = \frac{1}{k} - \frac{1}{k+1}$, we get

$$2007 \cdot (1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2006} - \frac{1}{2007}) = 2006.$$

17. Ans: 18063.

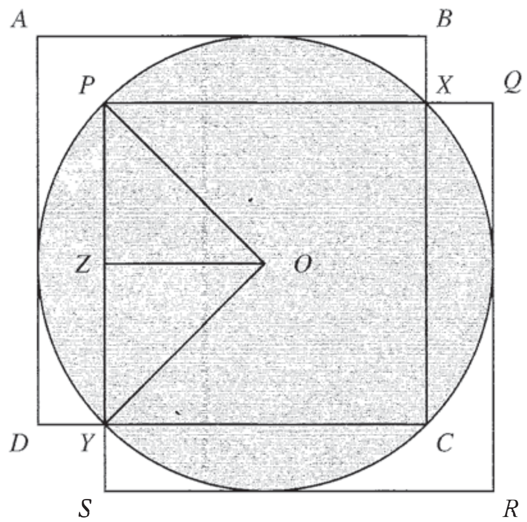
Since there is no carry, this is simply 9×2007. Or observe it is actually

$$222\underbrace{999999999\ldots999}_{2004\ 9's}777.$$

18. Ans: 1.

As the diagram below shows, $PY = \sqrt{2}OP$ and $YS = OP - \frac{1}{2}PY$. Thus,

$$PS = PY + YS = \frac{2+\sqrt{2}}{2}OP = \frac{2+\sqrt{2}}{2}(2-\sqrt{2}) = 1.$$



19. Ans: 286.

By assumption, $2007 - 5 = N \cdot k$ for some integer k. Factorize $2002 = 2 \cdot 7 \cdot 11 \cdot 13$. Since $200 < N < 300$, the only possibility is $N = 2 \cdot 11 \cdot 13 = 286$ and $k = 7$.

20. Ans: 1.

Rewrite the expression as

$$\frac{a^2+b^2+c^2-bc-ca-ab}{abc} = \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2abc}.$$

Since $a - b = -1$, $b - c = -1$ and $c - a = 2$, the result follows.

21. Ans: 50.

$$\angle BPC = 180° - (\angle PBC + \angle PCB) = 180° - \frac{2}{3}(\angle ABC + \angle ACB) = 180° - \frac{2}{3}120° = 100°.$$

Observe that PE bisects $\angle BPC$, the result follows.

22. Ans: 1.

We manipulate the expression and replace $x - y$ by 1 whenever necessary:

$$\begin{aligned} & x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4 \\ = \ & x^3(x-y) - y^3(x-y) - 3xy(x-y) \\ = \ & (x-y)(x^2 + xy + y^2) - 3xy \\ = \ & (x-y)^2 \\ = \ & 1. \end{aligned}$$

23. Ans: 3.

Since $2008^2 - 2007^2 = n^2 - m^2$, we have $4015 = (n+m)(n-m)$, i.e. $5 \cdot 11 \cdot 73 = (n+m)(n-m)$. There're four possibilities: $n + m = 4015, n - m = 1$; $n + m = 803, n - m = 5$; $n + m = 365, n - m = 11$ and $n + m = 73, n - m = 55$. But the first one $n = 2008$, $m = 2007$ is ruled out by assumption, the remaining pairs are $n = 404, m = 399$; $n = 188, m = 177$ and $n = 64, m = 9$.

24. Ans: 3.

Since $(x + \sqrt{xy} + y)(x - \sqrt{xy} + y) = (x+y)^2 - (\sqrt{xy})^2 = x^2 + xy + y^2$, we have $9 \cdot (x - \sqrt{xy} + y) = 27$. The result follows.

25. Ans: 075.

It suffices to make it divisible by 105 after appending. As $2007000 = 105 \times 19114 + 30$, the least number that we need to add is 75.

26. Ans: 12.

We need the maximal n such that $(n^3)^{2007} < 2007^{2007}$. We need $n^3 < 2007$. By calculation $12^3 = 1728 < 2007 < 2197 = 13^3$. The result follows.

27. Ans: 5.

Observe $x = \sqrt{(4 - \sqrt{3})^2} = 4 - \sqrt{3}$ and $x^2 - 8x + 13 = 19 - 8\sqrt{3} - 32 + 8\sqrt{3} + 13 = 0$. Use long division,

$$x^4 - 6x^3 - 2x^2 + 18x + 23 = (x^2 - 8x + 13)(x^2 + 2x + 1) + 10.$$

Thus the original expression equals

$$\frac{(x^2 - 8x + 13)(x^2 + 2x + 1) + 10}{x^2 - 8x + 13 + 2} = \frac{10}{2} = 5.$$

28. Ans: 2.

Using $ax + 1 = -x - a$, we have $x = -1$ or $a = -1$. But when $a = 1$, the original equation has no real roots. Thus $x = -1$, we have $a = 2$.

29. Ans: 45.

2007 is the 1004-th odd number. If 2007 is in the group $k + 1$, then $1 + 2 + \cdots + k < 1004 \leq 1 + 2 + \ldots (k + 1)$. Thus

$$\frac{k(k+1)}{2} < 1004 \leq \frac{(k+1)(k+2)}{2}.$$

We get $k = 44$. So 2007 is in group 45 which has 45 odd integers.

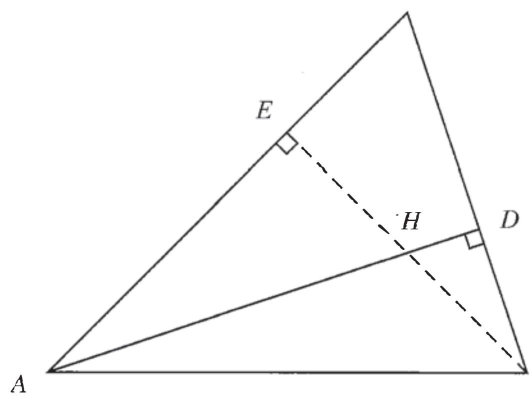
30. Ans: 15.

Construct CE which is perpendicular to AB where E is on AB. Since $\angle BAC = 45°$ (given), $AE = CE$. Thus the right $\triangle AEH$ is congruent to right $\triangle CEB$ So $AH = CB = 5$.

Next $\triangle ADB$ is similar to $\triangle CDH$, thus

$$\frac{BD}{AD} = \frac{HD}{CD} = \frac{AD - AH}{CD} \text{ which implies } \frac{3}{AD} = \frac{AD - 5}{2}.$$

we get $AD^2 - 5AD - 6 = 0$ Solving $AD = 6$ only as $AD > 0$. The result follows.



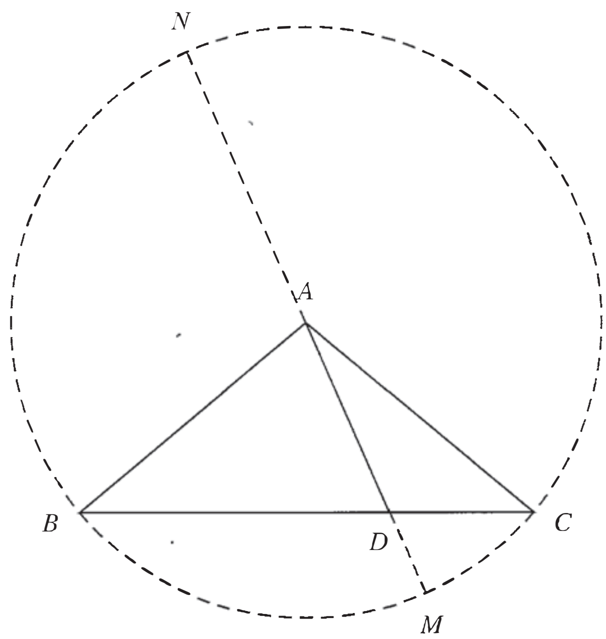
Note, if one is familiar with trigonometry, one may have an alternative solution as follows: Let α and β denote $\angle BAD$ and $\angle DAC$ respectively. Using $\tan(\alpha+\beta) = \tan 45^\circ = 1$, one get

$$\frac{\frac{3}{AD} + 2AD}{1 - \frac{3}{AD}\frac{2}{AD}} = 1,$$

consequently $AD = 6$.

31. Ans: 2.

Construct a circle with A as the center and $AB = AC = \sqrt{3}$ as the radius. Extend AD to meet the circumference at M and N as shown.



Using the Intersecting Chord Theorem

$$BD \cdot DC = MD \cdot ND = (\sqrt{3} - 1)(\sqrt{3} + 1) = 2.$$

32. Ans: 7.

Observe that for $n \geq 2$, 2^{2^n} always ends with a 6. The result follows.

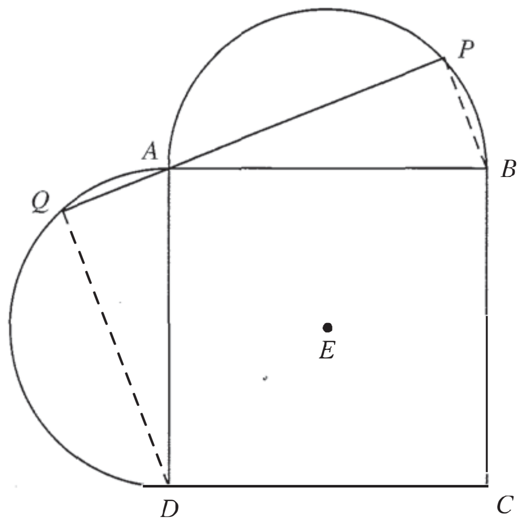
33. Ans: 34.

Join QD and PB, it is easy to show that right $\triangle DQA$ is congruent to right $\triangle APB$. Thus $PB = QA = 14$.

Apply Pythagoras' Theorem to $\triangle APB$,

$$(\sqrt{2}x)^2 = 46^2 + 14^2.$$

Solve, $x = 34$.



34. Ans: 37.

Since $247 = 13 \cdot 19$, one of $n, n+1, n+2$ is divisible by 13, call it a and one by 19 call it b. Clearly $|b - a| \leq 2$.

Let $b = 19c$. When $c = 1$, since $|b - a| \leq 2$, a is among $17, 18, 19, 20, 21$. But none is divisible by 13, hence we try $c = 2$. Now $b = 38$ and a is among $36, 37, 38, 39, 40$, hence $a = 39$. Thus the least $n = 37$.

35. Ans: 4265.

Let $N + 496 = a^2$ and $N + 224 = b^2$ for some positive integers a and b. Then $a^2 - b^2 = 496 = 224 = 272 = 2^4 \cdot 17$. Thus $17|(a+b)(a-b)$. If $17|a - b$ then $a - b \geq 17$ and $a + b \leq 16$, impossible. Thus $17|a + b$.

We have five possibilities for $(a + b, a - b)$: $(17, 16), (34, 8), (68, 4), (136, 2), (272, 1)$. Solve and discard non-integer solutions, we have $(a, b) = (21, 13), (36, 32)$ and $(69, 67)$. Thus the largest N is $69^2 - 496 = 4265$.