Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 1)

Wednesday, 30 May 2007 **0930-1200**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

1. Let A be any k-element subset of the set $\{1, 2, 3, 4, \cdots, 100\}$. Determine the minimum value of k such that we can always guarantee the existence of two numbers a and b in A such that $|a - b| \leq 4$.

2. Determine the number of those 0-1 binary sequences of ten 0's and ten 1's which do not contain three 0's together.

3. Let A be the set of any 20 points on the circumference of a circle. Joining any two points in A produces one chord of this circle. Suppose every three such chords are not concurrent. Find the number of regions within the circle which are divided by all these chords.

4. In each of the following 7-digit natural numbers:

$$1001011, 5550000, 3838383, 7777777,$$

every digit in the number appears at least 3 times. Find the number of such 7-digit natural numbers.

5. Let $A = \{1, 2, 3, 4, \ldots, 1000\}$. Let m be the number of 2-element subsets $\{a, b\}$ of A such that $a \times b$ is divisible by 6. Find the value of $\lfloor m/10 \rfloor$. (Here and in subsequent questions $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

6. Find the number of non-negative integer solutions of the following inequality:

$$x + y + z + u \leq 20.$$

7. In how many different ways can 7 different prizes be awarded to 5 students in such a way that each student has at least one prize?

8. Let ABC be any triangle. Let D and E be the points respectively in the segments of AB and BC such that $AD = 7DB$ and $BE = 10EC$. Assume that AE and CD meet at point F. Determine $\lfloor k \rfloor$, where k is the real number such that $AF = k \times FE$.

9. Let $S = \{1, 2, 3, 4, \cdots, 50\}$. A 3-element subset $\{a, b, c\}$ of S is said to be *good* if $a + b + c$ is divisible by 3. Determine the number of 3-elements of S which are good.

10. Let $x_1, x_2, \ldots, x_{1970}$ be positive integers satisfying $x_1 + x_2 + \cdots + x_{1970} = 2007$. Determine the largest possible value of $x_1^3 + x_2^3 + \cdots + x_{1970}^3$.

11. Determine the largest value of a such that a satisfies the equations $a^2 - bc - 8a + 7 = 0$ and $b^2 + c^2 + bc - 6a + 6 = 0$ for some real numbers b and c.

12. Determine the number of distinct integers among the numbers

$$\left\lfloor \frac{1^2}{2007} \right\rfloor, \quad \left\lfloor \frac{2^2}{2007} \right\rfloor, \quad \cdots \quad, \quad \left\lfloor \frac{2007^2}{2007} \right\rfloor.$$

13. Determine the number of pairs (a, b) of integers with $1 \le b < a \le 200$ such that the sum $(a + b) + (a - b) + ab + a/b$ is a square of a number.

14. This question has been deleted.

15. In an acute-angled triangle ABC, points D, E, and F are the feet of the perpendiculars from A, B, and C onto BC, AC and AB, respectively. Suppose $\sin A = \frac{3}{5}$ and $BC = 39$, find the length of AH, where H is the intersection AD with BE.

16. Let O be the centre of the circumcircle of $\triangle ABC$, P and Q the midpoints of AO and BC, respectively. Suppose $\angle CBA = 4\angle OPQ$ and $\angle ACB = 6\angle OPQ$. Find the size of $\angle OPQ$ in degrees.

17. In $\triangle ABC$, $AC > AB$, the internal angle bisector of $\angle A$ meets BC at D, and E is the foot of the perpendicular from B onto AD. Suppose $AB = 5$, $BE = 4$ and $AE = 3$. Find the value of the expression $\left(\frac{AC+AB}{AC-AB}\right) ED$.

18. Find the value of

$$\prod_{k=1}^{45} \tan(2k-1)^\circ.$$

19. Find the radius of the circle inscribed in a triangle of side lengths $50, 120, 130$.

20. Suppose that $0 < a < b < c < d = 2a$ and

$$(d-a)\left(\frac{a^2}{b-a} + \frac{b^2}{c-b} + \frac{c^2}{d-c}\right) = (a+b+c)^2.$$

Find bcd/a^3.

21. Let f be a function so that

$$f(x) - \frac{1}{2}f\left(\frac{1}{x}\right) = \log x$$

for all $x > 0$, where log denotes logarithm base 10. Find $f(1000)$.

22. Let O be an interior point of $\triangle ABC$. Extend AO to meet the side BC at D. Similarly, extend BO and CO to meet CA and AB respectively at E and F. If $AO = 30$, $FO = 20$, $BO = 60$, $DO = 10$ and $CO = 20$, find EO.

23. For each positive integer n, let a_n denote the number of n-digit integers formed by some or all of the digits 0, 1, 2, and 3 which contain neither a block of 12 nor a block of 21. Evaluate a_9.

24. Let S be any nonempty set of k integers. Find the smallest value of k for which there always exist two distinct integers x and y in S such that $x+y$ or $x-y$ is divisible by 2007.

25. Let P be a 40-sided convex polygon. Find the number of triangles S formed by the vertices of P such that any two vertices of S are separated by at least two other vertices of P.