Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 1 Solutions)

1. Ans: 21
   If A is the set of multiples of 5 in $\{1, 2, 3, \cdots, 100\}$, then $|A| = 20$ and $|a - b| \geq 5$ for every two numbers in A. Thus, if $|A| = 20$, the existence such two numbers cannot be guaranteed. However, if $|A| = 21$, by the pigeonhole principle, there must be two numbers a, b in one of the following twenty sets:
   $$\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10\}, \cdots, \{96, 97, 98, 99, 100\},$$
   and so $|a - b| \leq 4$. Thus the answer is 21.

2. Ans: 24068
   In such a binary sequence, 0's either appear singly or in blocks of 2. If the sequence has exactly m blocks of double 0's, then there are $10 - 2m$ single 0's. The number of such binary sequences is
   $$\binom{11}{m} \times \binom{11-m}{10-2m}.$$
   Thus the answer is
   $$\sum_{m=0}^{5} \binom{11}{m} \times \binom{11-m}{10-2m} = 24068.$$

3. Ans: 5036
   Consider such a figure as a plane graph G. Then the answer is equal to the number of interior faces of G. The number of vertices in G is $20 + \binom{20}{4}$. The sum of all degrees is
   $$20 \times 21 + 4 \times \binom{20}{4}$$
   and so the number of edges in G is
   $$10 \times 21 + 2 \times \binom{20}{4}.$$
   Hence the number of interior faces, by Euler's formula, is
   $$\binom{20}{4} + 10 \times 21 - 20 + 1 = 5036.$$

Second Solution: Let P_n be the number of such regions with n points on the circumference. Then $P_1 = 1$, $P_2 = 2$ and in general, for $n \geq 2$, $P_{n+1} = P_n + n + \sum_{i=1}^{n-2} i(n-i)$. This can be obtained as follows. Suppose there are $n+1$ points $a_0, \ldots, a_n$ in that order on the circumference. The chords formed by $a_1, \ldots, a_n$ create P_n regions. The chord a_0a_1 adds one region. The chord a_0a_1 adds $1 + 1 \times (n-2)$ regions as this chord intersects the existing chords in $1 \times (n-2)$ points. Similarly, the chord a_0a_2 adds $1 + 2 \times (n-3)$ regions, etc. From this, it is easy to show that

$$\begin{aligned} P_n &= 1 + \binom{n}{2} + \binom{n-2}{2} + 2\binom{n-3}{2} + \cdots + (n-3)\binom{2}{2} \\ &= 1 + \binom{n}{2} + \binom{n-1}{3} + 1\binom{n-3}{2} + \cdots + (n-4)\binom{2}{2} \\ &= \cdots = 1 + \binom{n}{2} + \binom{n-1}{3} + \binom{n-2}{3} + \cdots + \binom{3}{3} = 1 + \binom{n}{2} + \binom{n}{4} \end{aligned}$$

Thus $P_{20} = 1 + \binom{20}{2} + \binom{20}{4} = 5036$.

4. Ans: 2844

If only one digit appears, then there are 9 such numbers. If the two digits that appear are both nonzero, then the number of such numbers is

$$2 \times \binom{7}{3}\binom{9}{2} = 2520.$$

If one of two digits that appear is 0, then the number of such numbers is

$$\left(\binom{6}{4} + \binom{6}{3}\right) \times \binom{9}{1} = 315.$$

Hence the answer is $9 + 2520 + 315 = 2844$.

5. Ans: 20791

Let

$$A_6 = \{k \in A : 6 \mid k\}; \quad A_2 = \{k \in A : 2 \mid k, 6 \nmid k\}; \quad A_3 = \{k \in A : 3 \mid k, 6 \nmid k\}.$$

Note that

$$\begin{aligned} |A_6| &= \left\lfloor \frac{1000}{6} \right\rfloor = 166; \qquad |A_2| = \left\lfloor \frac{1000}{2} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor = 334; \\ |A_3| &= \left\lfloor \frac{1000}{3} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor = 167. \end{aligned}$$

For the product $a \times b$ to be divisible by 6, either (i) one or both of them are in A_6 or (ii) one is in A_2 and the other is in A_3. Hence

$$m = \binom{166}{2} + 166 \times (1000 - 166) + 334 \times 167 = 207917.$$

6. Ans: 10626
Let $v = 20 - (x + y + z + u)$. Then $v \geq 0$ if and only if $x + y + z + u \leq 20$. Hence the answer is equal to the number of non-negative integer solutions of the following equation:

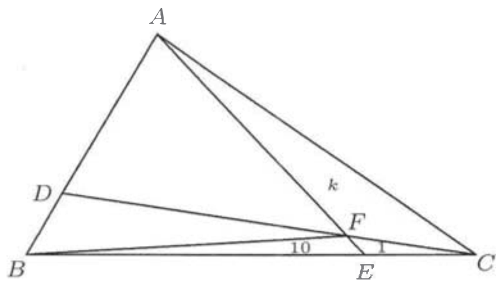
$$x + y + z + u + v = 20$$

and the answer is $\binom{24}{4} = 10626$.

7. Ans: 16800
Either one student receives three prizes; or two students are each awarded two prizes. Thus the answer is

$$\binom{7}{3} \times 5! + \binom{7}{2} \times \binom{5}{2} \times \frac{1}{2} \times 5! = 16800.$$

8. Ans: 77
Assume that $[CEF] = 1$. Then, $[AFC] = k$ and $[BFC] = 11$. Since $AD = 7DB$, $k = [AFC] = 7[BFC] = 77$.



9. Ans: 6544
For $i = 0, 1, 2$, let

$$A_i := \{1 \leq k \leq 50 : 3 \mid (k - i)\}.$$

Then $|A_0| = 16$, $|A_1| = 17$ and $|A_2| = 17$. It can be shown that $3 \mid (a+b+c)$ if and only if either $\{a, b, c\} \subseteq A_i$ for some i or $\{a, b, c\} \cap A_i \neq \emptyset$ for all $i = 1, 2, 3$. Thus the answer is

$$\binom{16}{3} + \binom{17}{3} + \binom{17}{3} \quad 16 \times 17 \times 17 = 6544.$$

10. Ans: 56841
We may assume $x_1 \le x_2 \le \cdots \le x_{1970}$. For $1 \le x_i \le x_j$ with $i \le j$, we have $x_i^3 + x_j^3 \le x_i^3 + x_j^3 + 3(x_j - x_i)(x_j + x_i - 1) = (x_i - 1)^3 + (x_j + 1)^3$. Thus when $x_1 = x_2 = \cdots = x_{1969} = 1$ and $x_{1970} = 38$, the expression $x_1^3 + x_2^3 + \cdots + x_{1970}^3$ attains its maximum value of $1969 + 38^3 = 56841$.

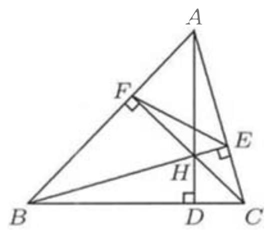
11. Ans: 9
Substituting the first equation $bc = a^2 - 8a + 7$ into the second equation, we have $(b+c)^2 = (a-1)^2$ so that $b + c = \pm(a-1)$. That means b and c are roots of the quadratic equation $x^2 \mp (a-1)x + (a^2 - 8a + 7) = 0$. Thus its discriminant $\triangle = [\mp(a-1)]^2 - 4(a^2 - 8a + 7) \ge 0$, or equivalently, $1 \le a \le 9$. For $b = c = 4$, $a = 9$ satisfies the two equations. Thus the largest value of a is 9.

12. Ans: 1506
Let $a_i = \lfloor i^2/2007 \rfloor$. Note that $\frac{(n+1)^2}{2007} - \frac{n^2}{2007} = \frac{2n+1}{2007} \le 1$ if and only if $n \le 2006/2 = 1003$. Since, $a_1 = 0$, and $a_{1003} = 501$, we see that $a_1, \ldots, a_{1003}$ assume all the values from 0 to 501. We also conclude that $a_{1004}, \ldots, a_{2007}$ are mutually distinct integers. Therefore, the answer is $502 + 1004 = 1506$.

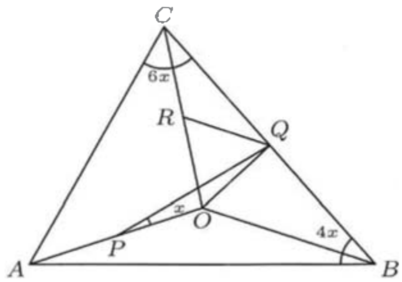
13. Ans: 112
Since the number $(a+b) + (a-b) + ab + a/b = (a/b)(b+1)^2$ is a perfect square with b and $b+1$ relatively prime, the number a/b must be a perfect square. Let $a/b = n^2$. As $a > b$, the number $n \ge 2$ so that $a = bn^2 \le 200$. From this, a can be determined once b and n are chosen. Hence, it suffices to count the number of pairs of (b, n) satisfying $bn^2 \le 200$ with $b \ge 1$ and $n \ge 2$. Hence the answer is $\lfloor 200/2^2 \rfloor + \cdots + \lfloor 200/14^2 \rfloor = 112$.

14. This question has been deleted.

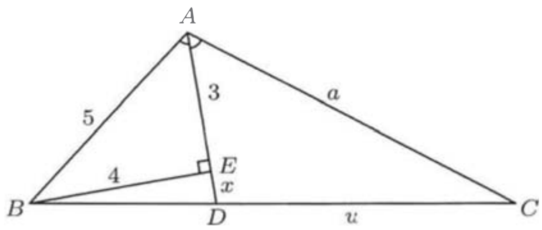
15. Ans: 52
First we know that $AE = AB\cos A$ and $AF = AC\cos A$. By cosine rule, $EF^2 = AE^2 + AF^2 - 2AE \times AF\cos A = \cos^2 A(AB^2 + AC^2 - 2AB \times AC\cos A) = BC^2\cos^2 A$. Therefore $EF = BC\cos A$. It is easy to see that A, E, H, F lie on a circle with diameter AH. Thus $AH = \frac{EF}{\sin A} = BC\cot A = 39 \times \frac{4}{3} = 52$.



16. Ans: 12
Let R be the midpoint of OC. Then $PO = RO = RQ$. Let $\angle OPQ = x$. Then $\angle POR = 2\angle B = 8x$ and $\angle BOA = 2\angle C = 12x$. Also, $\angle ROQ = \angle A = 180° - 10x$. It follows that $\angle PQO = 180° - x - 8x - (180° - 10x) = x$ and therefore $PO = OQ$. Thus $\triangle OQR$ is equilateral, so $\angle ROQ = 180° - 10x = 60°$ and $x = 12°$.



17. Ans: 3
Let $DE = x$, $AC = a > 5$ and $CD = u$. Then $BD = \sqrt{4^2 + x^2}$ and $\cos\angle CAD = \cos\angle BAD = 3/5$. By the cosine rule applied to $\triangle ADC$, we have $u^2 = (3+x)^2 + a^2 - 2a(3+x)(3/5)$. Using the angle bisector theorem, we have $\frac{u^2}{4^2+x^2} = \frac{a^2}{5^2}$. Thus $25[(3+x)^2 + a^2 - 2a(3+x)(3/5)] = a^2(16+x^2)$.



This can be simplified to $(a^2-25)x^2 + 30(a-5)x - 9(a-5)^2 = 0$. Since $a > 5$, we can cancel a common factor $(a-5)$ to get $(a+5)x^2 + 30x - 9(a-5) = 0$, or equivalently $(x+3)((a+5)x - 3(a-5)) = 0$. Thus $x = 3(a-5)/(a+5)$. From this, we obtain $\left(\frac{AC+AB}{AC-AB}\right) ED = \left(\frac{a+5}{a-5}\right) x = 3$.

18. Ans: 1
Note that $\tan(90° - \theta) = 1/\tan\theta$ for $0° < \theta < 90°$and that the product is positive. Setting $j = 46 - k$,

$$\prod_{k=1}^{45} \tan(2k-1)° = \prod_{j=1}^{45} \tan(90 - (2j-1))° = \frac{1}{\prod_{j=1}^{45} \tan(2j-1)°} = 1.$$

19. Ans: 20
The triangle is a right triangle with area $A = 5 \times 120/2 = 300$. The semiperimeter is $s = \frac{1}{2} \times (50 + 120 + 130) = 150$. Hence the inradius is $A/s = 20$.

20. Ans: 4
Set

$$\mathbf{u} = (\sqrt{b-a}, \sqrt{c-b}, \sqrt{d-c})$$

$$\text{and} \quad \mathbf{v} = \Big(\frac{a}{\sqrt{b-a}}, \frac{b}{\sqrt{c-b}}, \frac{c}{\sqrt{d-c}}\Big).$$

Then $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|$ and hence $\mathbf{u} = \alpha \mathbf{v}$ for some $\alpha \geq 0$. Thus

$$\frac{b-a}{a} = \frac{c-b}{b} = \frac{d-c}{c}.$$

Hence $b/a = c/b = d/c$, i.e., a, b, c, d is a geometric progression $a, ra, r^2a, r^3a = 2a$. Now $bcd/a^3 = r^6 = (r^3)^2 = 4$.

21. Ans: 2
Putting $1/x$ in place of x in the given equation yields

$$f\Big(\frac{1}{x}\Big) - \frac{1}{2} f(x) = -\log x.$$

Solving for $f(x)$, we obtain $f(x) = \frac{2}{3} \log x$. Hence $f(1000) = 2$.

22. Ans: 20
We have

$$\frac{OD}{AD} = \frac{[BOC]}{[ABC]}, \quad \frac{OE}{BE} = \frac{[AOC]}{[ABC]}, \quad \frac{OF}{CF} = \frac{[AOB]}{[ABC]}.$$

Thus

$$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1 \quad \Rightarrow \quad OE = 20.$$

23. Ans: 53808
For $n \geq 3$, among the a_n such integers, let b_n denote the number of those that end with 1. By symmetry, the number of those that end with 2 is also equal to b_n. Also the number of those that end with 0 or 3 are both a_{n-1}. Thus

$$a_n = 2a_{n-1} + 2b_n.$$

Among the b_n integers that end with 1, the number of those that end with 11 is b_{n-1} while the number of those that end with 01 or 31 are both a_{n-2}. Thus

$$b_n = b_{n-1} + 2a_{n-2}.$$

Solving, we get $a_n = 3a_{n-1} + 2a_{n-2}$. Since $a_1 = 3$ and $a_2 = 10$, we get $a_9 = 73368$.

24. Ans: 1005
Partition all the possible remainders when divided by 2006 as follows:

$$(0),\ (1, 2006),\ (2, 3005),\ \ldots\ ,\ (1003, 1004).$$

Suppose S has 1005 elements. If S has two elements with their difference divisible by 2007, we are done. Otherwise the elements of S have distinct remainders when divided by 2007. By the pigeonhole principle, S has two integers whose sum is divisible by 2007. The set $\{1, 2, \ldots, 1004\}$ does not have 2 elements, x, y such that $x + y$ or $x - y$ is divisible by 2007.

25. Ans: 7040
For better understanding, we consider the general case when P has n vertices, where $n \geq 9$. We first count the number of such triangles S having a particular vertex A. The number is $\binom{n-7}{2}$. (This can be obtained as follows. Let the triangle be ABC ordered clockwise. Then B has a "left" neighbour and C has a "right" neighbour. The location of of B and C are uniquely determined by their neighbours. Besides A, B, C and the two vertices to the left and two vertices to the right of A, the two neighbours can be chosen from the remaining $n - 7$ vertices. Thus the required answer is $\frac{n}{3}\binom{n-7}{2} = 7040$.