Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2020
(Open Section, Round 1)

Tuesday, 8 September 2020 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

In this paper, let $\mathbb{R}$ denote the set of all real numbers, $\lfloor x \rfloor$ denote the greatest integer not exceeding x and let $\lceil x \rceil$ denote the smallest integer not less than x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$; $\lceil 5 \rceil = 5$, $\lceil 2.8 \rceil = 3$, and $\lceil -2.3 \rceil = -2$.

1. If S is the sum of all the *real* roots of the equation $x^2 + \dfrac{1}{x^2} = 2020^2 + \dfrac{1}{2020^2}$, find $\lfloor S \rfloor$.

2. Find the largest positive integer x that satisfies the equation
$$(\lfloor x \rfloor - 2020)^2 + (\lceil x \rceil - 2030)^2 = (\lfloor x \rfloor - \lceil x \rceil + 10)^2 .$$
(Note: If you think that the above equation has no solution in positive integers, enter your answer as "0".)

3. Let $S_n = \dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + \cdots + \dfrac{1}{(2n-1) \times (2n+1)}$. Find the value of n such that S_n takes the value of 0.48.

4. Given that the three planes in the Cartesian space with equations $2x + 4y + 6z = 5$, $3x + 5y + 2z = 6$ and $8x + 14y + az = b$ have a common line of intersection, find the value of $a + b$.

5. Let i be the complex number $\sqrt{-1}$, and n be the smallest positive integer such that $(\sqrt{3} + i)^n = a$, where a is a real number. Find the value of $\lfloor n - a \rfloor$.

6. In the three-dimensional Cartesian space, let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote unit vectors along three mutually perpendicular directions x, y and z-axes respectively. Three straight lines l_1, l_2 and l_3 have equations defined by
$$l_1: \quad \mathbf{r} = (4 + \lambda)\mathbf{i} + (5 + \lambda)\mathbf{j} + (6 + \lambda)\mathbf{k},$$
$$l_2: \quad \mathbf{r} = (4 + 3\mu)\mathbf{i} + (5 - \mu)\mathbf{j} + (6 - 2\mu)\mathbf{k},$$
$$l_3: \quad \mathbf{r} = (1 + 6\nu)\mathbf{i} + (2 + 2\nu)\mathbf{j} + (3 + \nu)\mathbf{k},$$
where μ, λ and ν are real numbers. If the area of the triangle enclosed by the three lines l_1, l_2 and l_3 is denoted by S, find the value of $10S^2$.

7. Given that $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(a^2 - b^2) = (a - b)\left(f(a) + f(b)\right)$$
for all real numbers a and b, and that $f(1) = \dfrac{1}{101}$. Find the value of $\displaystyle\sum_{k=1}^{100} f(k)$.

8. Find the sum of all the positive integers n such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square.

   (Note: If you think that there are infinitely many positive integers n that satisfy the above conditions, enter your answer as "9999".)

9. Assume that

$$(x+2+m)^{2019} = a_0 + a_1(x+1) + a_2(x+1)^2 + \cdots + a_{2019}(x+1)^{2019}.$$

Find the largest possible integer m such that

$$(a_0 + a_2 + a_4 + \cdots + a_{2018})^2 - (a_1 + a_3 + a_5 + \cdots + a_{2019})^2 \leq 2020^{2019}.$$

10. Given that $S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n(n+k)}}$. Find the value of $\lfloor (S+2)^2 \rfloor$.

11. Let $A = \{1, 2, \cdots, 10\}$. Count the number of ordered pairs (S_1, S_2), where S_1 and S_2 are non-intersecting and non-empty subsets of A such that the largest number in S_1 is smaller than the smallest number in S_2. For example, if $S_1 = \{1, 4\}$ and $S_2 = \{5, 6, 7\}$, then (S_1, S_2) is such an ordered pair.

12. Each cell of an 2020×2020 table is filled with a number which is either 1 or -1. For $i = 1, \ldots, 2020$, let R_i be the product of all the numbers in the ith row and let C_i be the product of all the numbers in the ith column. Suppose $R_i + C_i = 0$ for all $i = 1, \ldots, 2020$. What is the least number of -1's in the table?

13. Assume that the sequence $\{a_k\}_{k=1}^{\infty}$ follows an arithmetic progression with $a_2 + a_4 + a_9 = 24$. Find the maximum value of $S_8 \times S_{10}$, where S_k denotes the sum $a_1 + a_2 + \cdots + a_k$.

14. Consider all functions $g : \mathbb{R} \to \mathbb{R}$ satisfying the conditions that

    (i) $|g(a) - g(b)| \leq |a - b|$ for any $a, b \in \mathbb{R}$;

    (ii) $g(g(g(0))) = 0$.

    Find the *largest* possible value of $g(0)$.

15. A sequence $\{a_i\}_{i=1}^{\infty}$ is called a *good* sequence if $\frac{S_{2n}}{S_n}$ is a constant for all $n \geq 1$, where S_k denotes the sum $a_1 + a_2 + \cdots + a_k$. Suppose it is known that the sequence $\{a_i\}_{i=1}^{\infty}$ is a *good* sequence that follows an arithmetic progression. Determine a_{2020} if $a_1 = 1 \neq a_2$.

16. Determine the smallest positive integer p such that the system

$$\begin{cases} 6x + 4y + 3z = 0 \\ 4xy + 2yz + pxz = 0 \end{cases}$$

has more than one set of real solutions in x, y, z.

17. Let ABC be a triangle with $a = BC, b = AC$ and $c = AB$. It is given that $c = 100$ and

$$\frac{\cos A}{\cos B} = \frac{b}{a} = \frac{4}{3}.$$

Let P be a point on the inscribed circle of $\triangle ABC$. Find the maximum value of

$$PA^2 + PB^2 + PC^2.$$

18. Find the largest positive integer n less than 2020 such that $\binom{n-1}{k} - (-1)^k$ is divisible by n for $k = 0, 1, \ldots, n-1$.

19. Assume that $\{a_k\}_{k=1}^{\infty}$ is a sequence with the property that for any distinct positive integers m, n, p, q with $m + n = p + q$, the following equality always holds:

$$\frac{a_m + a_n}{(a_m + 1)(a_n + 1)} = \frac{a_p + a_q}{(a_p + 1)(a_q + 1)}.$$

Given $a_1 = 0$ and $a_2 = \frac{1}{2}$, determine $\frac{1}{1 - a_5}$.

(Hint: Consider $c_k = \frac{1}{a_k + 1} - \frac{1}{2}$ for all positive integer k.)

20. In the triangle ABC, the incircle touches the sides BC, CA, AB at D, E, F respectively. The line segments ED and AB are extended to intersect at P such that $AB = BP = PD$. Suppose $CA = 9$. Find the value of $(ABC)^2$, where (ABC) is the area of the triangle ABC.

21. In an acute-angled triangle ABC, $AB = 75, AC = 53$, the external bisector of $\angle A$ on CA produced meets the circumcircle of triangle ABC at E, and F is the foot of the perpendicular from E onto AB. Find the value of $AF \times FB$.

22. Let $\{a_k\}_{k=1}^{\infty}$ be an increasing sequence with $a_k < a_{k+1}$ for all $k = 1, 2, 3, \cdots$ formed by arranging all the terms in the set $\{2^r + 2^s + 2^t : 0 \le r < s < t\}$ in increasing order. Find the largest value of the integer n such that $a_n \le 2020$.

23. Let n be a positive integer and S be the set of all numbers that can be written in the form $\sum_{i=2}^{k} a_{i-1}a_i$ with $a_1, \ldots, a_k$ being positive integers that sum to n. Suppose the average value of all the numbers in S is 88199. Determine n.

24. Let x, y, z and w be real numbers such that $x + y + z + w = 5$. Find the minimum value of $(x + 5)^2 + (y + 10)^2 + (z + 20)^2 + (w + 40)^2$.

25. Let p and q be positive integers satisfying the equation $p^2 + q^2 = 3994(p - q)$. Determine the largest possible value of q.