Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2011
(Open Section, Round 1)

Wednesday, 1 June 2011 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Throughout this paper, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$ (This notation is used in Questions 7, 9, 19 and 20).

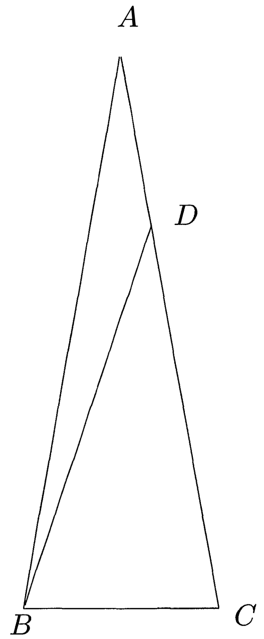
1. A circular coin A is rolled, without sliding, along the circumference of another stationary circular coin B with radius twice the radius of coin A. Let x be the number of degrees that the coin A makes around its centre until it first returns to its initial position. Find the value of x.

2. Three towns X, Y and Z lie on a plane with coordinates $(0,0)$, $(200,0)$ and $(0,300)$ respectively. There are 100, 200 and 300 students in towns X, Y and Z respectively. A school is to be built on a grid point (x,y), where x and y are both integers, such that the overall distance travelled by all the students is minimized. Find the value of $x+y$.

3. Find the last non-zero digit in 30!.

   (For example, $5! = 120$; the last non-zero digit is 2.)

4. The diagram below shows $\triangle ABC$, which is isoceles with $AB = AC$ and $\angle A = 20^\circ$. The point D lies on AC such that $AD = BC$. The segment BD is constructed as shown. Determine $\angle ABD$ in degrees.



5. Given that $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$, evaluate $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha}$.

6. The number 25 is expressed as the sum of positive integers $x_1, x_2, \cdots, x_k$, where $k \leq 25$. What is the maximum value of the product of $x_1, x_2, x_3, \cdots$, and x_k?

7. Let x_0 be the largest (real) root of the equation $x^4 - 16x - 12 = 0$. Evaluate $\lfloor 10x_0 \rfloor$.

8. Let $x_i \in \{\sqrt{2}-1, \sqrt{2}+1\}$, where $i = 1, 2, 3, \cdots, 2012$. Define

$$S = x_1x_2 + x_3x_4 + x_5x_6 + \cdots + x_{2009}x_{2010} + x_{2011}x_{2012}.$$

How many different positive integer values can S attain?

9. Let A be the set of real numbers x satisfying the inequality $x^2 + x - 110 < 0$ and B be the set of real numbers x satisfying the inequality $x^2 + 10x - 96 < 0$. Suppose that the set of integer solutions of the inequality $x^2 + ax + b < 0$ is exactly the set of integers contained in $A \cap B$. Find the maximum value of $\lfloor |a - b| \rfloor$.

10. Given that

$$\begin{aligned} \alpha + \beta + \gamma &= 14 \\ \alpha^2 + \beta^2 + \gamma^2 &= 84 \\ \alpha^3 + \beta^3 + \gamma^3 &= 584, \end{aligned}$$

find $\max\{\alpha, \beta, \gamma\}$.

11. Determine the largest even positive integer which cannot be expressed as the sum of two composite odd positive integers.

12. Let a, b, c be positive integers such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ and $\gcd(a, b, c) = 1$. Suppose $a + b \leq 2011$. Determine the largest possible value of $a + b$.

13. Let $x[n]$ denote $x^{x^{\cdot^{\cdot^{x}}}}$, where there are n terms of x. What is the minimum value of n such that $9[9] < 3[n]$?

(For example, $3[2] = 3^3 = 27$; $2[3] = 2^{2^2} = 16$.)

14. In the triangle ABC, $\angle B = 90°$, $\angle C = 20°$, D and E are points on BC such that $\angle ADC = 140°$ and $\angle AEC = 150°$. Suppose $AD = 10$. Find $BD \cdot CE$.

15. Let $S = \{1, 2, 3, \cdots, 65\}$. Find the number of 3-element subsets $\{a_1, a_2, a_3\}$ of S such that $a_i \leq a_{i+1} - (i + 2)$ for $i = 1, 2$.

16. Determine the value of

$$\frac{3}{\sin^2 20°} - \frac{1}{\cos^2 20°} + 64 \sin^2 20°.$$

17. A real-valued function f satisfies the relation

$$\mathrm{f}(x^2 + x) + 2\mathrm{f}(x^2 - 3x + 2) = 9x^2 - 15x$$

for all real values of x. Find f(2011).

18. A collection of 2011 circles divide the plane into N regions in such a way that any pair of circles intersects at two points and no point lies on three circles. Find the last four digits of N.

19. If a positive integer N can be expressed as $\lfloor x\rfloor + \lfloor 2x\rfloor + \lfloor 3x\rfloor$ for some real numbers x, then we say that N is "visible"; otherwise, we say that N is "invisible". For example, 8 is visible since $8 = \lfloor 1.5\rfloor + \lfloor 2(1.5)\rfloor + \lfloor 3(1.5)\rfloor$, whereas 10 is invisible. If we arrange all the "invisible" positive integers in increasing order, find the 2011^{th} "invisible" integer.

20. Let A be the sum of all non-negative integers n satisfying

$$\lfloor \frac{n}{27} \rfloor = \lfloor \frac{n}{28} \rfloor.$$

Determine A.

21. A triangle whose angles are $A,\ B,\ C$ satisfies the following conditions

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \frac{12}{7},$$

and

$$\sin A \sin B \sin C = \frac{12}{25}.$$

Given that $\sin C$ takes on three possible values $s_1,\ s_2$ and s_3, find the value of $100 s_1 s_2 s_3$.

22. Let $x > 1,\ y > 1$ and $z > 1$ be positive integers for which the following equation

$$1! + 2! + 3! + \ldots + x! = y^z$$

is satisfied. Find the largest possible value of $x + y + z$.

23. Let ABC be a non-isosceles acute-angled triangle with circumcentre O, orthocentre H and $\angle C = 41°$. Suppose the bisector of $\angle A$ passes through the midpoint M of OH. Find $\angle HAO$ in degrees.

24. The circle γ_1 centred at O_1 intersects the circle γ_2 centred at O_2 at two points P and Q. The tangent to γ_2 at P intersects γ_1 at the point A and the tangent to γ_1 at P intersects $\gamma_2$at the point B where A and B are distinct from P. Suppose $PQ \cdot O_1O_2 = PO_1 \cdot PO_2$ and $\angle APB$ is acute. Determine the size of $\angle APB$ in degrees.

25. Determine

$$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{\binom{n}{i}}.$$

(Note: Here $\binom{n}{i}$ denotes $\dfrac{n!}{i!(n-i)!}$ for $i = 0, 1, 2, 3, \cdots, n$.)