Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2011
(Open Section, Round 1 Solution)

1. **Answer.** 1080

   Solution. The number of complete revolutions the first coin A has turned through is the sum of two components: the number of revolutions round the stationary coin B if A were *sliding* on B and the number of revolutions round A's own axis (perpendicular to its plane and through its centre) determined by the distance travelled on the circumference of B. Thus, the total number of revolutions is

$$1 + \frac{2\pi(2r)}{2\pi r} = 3.$$

   Hence the number of degrees $= 3 \times 360 = 1080$. □

2. **Answer.** 300

   Solution. We claim that the school must be built in Z. Suppose the school is to be built at another point A. The change in distance travelled

$$\begin{aligned} &= 300ZA + 200YA + 100XA - 200YZ - 100XZ \\ &= 100(ZA + AX - ZX) + 200(ZA + AY - ZY) \\ &> 0 \end{aligned}$$

   by triangle inequality. Thus, $\min(x + y) = 0 + 300 = 300$. □

3. **Answer.** 8

   Solution. We first obtain the prime factorization of 30!. Observe that 29 is the largest prime number less than 30. We have

$$\left\lfloor \frac{30}{2} \right\rfloor + \left\lfloor \frac{30}{2^2} \right\rfloor + \left\lfloor \frac{20}{2^3} \right\rfloor + \left\lfloor \frac{30}{2^4} \right\rfloor = 26$$

$$\left\lfloor \frac{30}{3} \right\rfloor + \left\lfloor \frac{30}{3^2} \right\rfloor + \left\lfloor \frac{30}{3^3} \right\rfloor = 14$$

$$\left\lfloor \frac{30}{5} \right\rfloor + \left\lfloor \frac{30}{5^2} \right\rfloor = 7$$

$$\left\lfloor \frac{30}{7} \right\rfloor = 4$$

$$\left\lfloor \frac{30}{11} \right\rfloor = 2$$

$$\left\lfloor \frac{30}{13} \right\rfloor = 2$$

$$\left\lfloor \frac{30}{17} \right\rfloor = \left\lfloor \frac{30}{19} \right\rfloor = \left\lfloor \frac{30}{23} \right\rfloor = \left\lfloor \frac{30}{29} \right\rfloor = 1.$$

Thus,

$$\begin{aligned}
30! &= 2^{26}\cdot 3^{14}\cdot 5^7\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 29\\
\frac{30!}{10^7} &= 2^{19}\cdot 3^{14}\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 29\\
&= 6^{14}\cdot 2^5\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 29\\
&\equiv 6(2)(1)(1)(9)(7)(9)(3)(9) (\text{mod } 10)\\
&\equiv 2(-1)(-3)(-1)(3)(-1) (\text{mod } 10)\\
&\equiv 8 (\text{mod } 10),
\end{aligned}$$

showing that the last non-zero digit is 8.

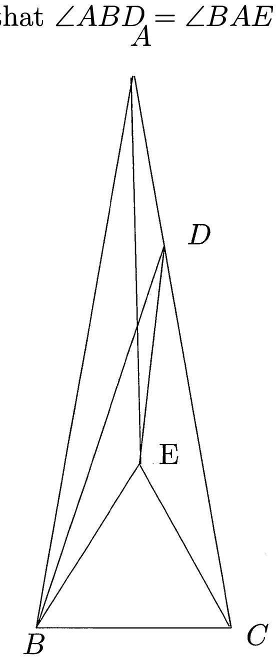
□

4. **Answer.** 10

Solution. Let E be the point inside $\triangle ABC$ such that $\triangle EBC$ is equilateral. Connect A and D to E respectively.

It is clear that $\triangle AEB$ and $\triangle AEC$ are congruent, since $AE = AE$, $AB = AC$ and $BE = CE$. It implies that $\angle BAE = \angle CAE = 10^o$.

Since $AD = BC = BE$, $\angle EBA = \angle DAB = 20^o$ and $AB = BA$, we have $\triangle ABE$ and $\triangle BAD$ are congruent, implying that $\angle ABD = \angle BAE = 10^o$.



□

5. **Answer.** 1

Solution. Since $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$, set $\cos\theta = \dfrac{\cos^2\alpha}{\cos\beta}$ and $\sin\theta = \dfrac{\sin^2\alpha}{\sin\beta}$. Then

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos^2\alpha + \sin^2\alpha = 1.$$

and so

$$\theta - \alpha = 2k\pi \text{ for some } k \in \mathbb{Z}.$$

Thus $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha$. Consequently,

$$\frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = \cos^2\beta + \sin^2\beta = 1.$$

□

6. **Answer.** 8748

Solution. Clearly, no x_i should be 1. If $x_i \geq 4$, then splitting it into two factors 2 and $x_i - 2$ will give a product of $2x_i - 4$ which is at least as large as x_i. Further, $3\times 3 > 2\times 2\times 2$, so any three factors of 2 should be replaced by two factors of 3. Thus, split 25 into factors of 3, retaining two 2's, which means $25 = 7\times 3 + 2\times 2$. The maximum product is thus $3^7 2^2 = 8748$. □

7. **Answer.** 27

Solution. Since $x^4 - 16x - 12 \equiv x^4 + 4x^2 + 4 - 4(x^2 + 4x + 4) \equiv (x^2 - 2x - 2)(x^2 + 2x + 6)$, we conclude that $x_0 = 1 + \sqrt{3}$ and so $1 + \sqrt{2.89} < x_0 < 1 + \sqrt{3.24}$. Consequently, $\lfloor 10x_0 \rfloor = 27$. □

8. **Answer.** 504

Solution. Note that $(\sqrt{2}-1)^2 = 3 - 2\sqrt{2}$, $(\sqrt{2}+1)^2 = 3 + 2\sqrt{2}$ and $(\sqrt{2}-1)(\sqrt{2}+1) = 1$. There are 1006 pairs of products in S; each pair of the product can be either $3-2\sqrt{2}, 3+2\sqrt{2}$ or 1. Let a be the number of these products with value $3 - 2\sqrt{2}$, b be the number of these products with value $3+2\sqrt{2}$ and c be the number of them with value 1. The $a+b+c = 1006$. Hence

$$S = a(3 - 2\sqrt{2}) + b(3 + 2\sqrt{2}) + c = 3a + 3b + c + 2\sqrt{2}(b - a).$$

For S to be a positive integer, $b = a$ and thus $2a + c = 1006$. Further,

$$S = 6a + c = 6a + 1006 - 2a = 4a + 1006.$$

From $2a + c = 1006$ and that $0 \leq a \leq 503$, it is clear that S can have 504 different positive integer values. □

9. **Answer.** 71

Note that $x^2 + x - 110 = (x - 10)(x + 11)$. Thus the set of real numbers x satisfying the inequality $x^2 + x - 110 < 0$ is $-11 < x < 10$.

Also note that $x^2 + 10x - 96 = (x - 6)(x + 16)$. Thus the set of real numbers x satisfying the inequality $x^2 + 10x - 96 < 0$ is $-16 < x < 6$.

Thus $A = \{x : -11 < x < 10\}$ and $B = \{x : -16 < x < 6\}$, implying that

$$A \cap B = \{x : -11 < x < 6\}.$$

Now let $x^2 + ax + b = (x - x_1)(x - x_2)$, where $x_1 \le x_2$. Then the set of integer solutions of $x^2 + ax + b < 0$ is

$$\{k : k \text{ is an integer}, x_1 < k < x_2\}.$$

By the given condition,

$$\{k : k \text{ is an integer}, x_1 < k < x_2\} = \{k : k \text{ is an integer}, -11 < k < 6\}$$
$$= \{-10, -9, \cdots, 5\}.$$

Thus $-11 \le x_1 < -10$ and $5 < x_2 \le 6$. It implies that $-6 < x_1 + x_2 < -4$ and $-66 \le x_1 x_2 < -50$.

From $x^2+ax+b = (x-x_1)(x-x_2)$, we have $a = -(x_1+x_2)$ and $b = x_1x_2$. Thus $4 < a < 6$ and $-66 \le b < -50$. It follows that $54 < a - b < 72$.

Thus $\max\lfloor|a - b|\rfloor \le 71$.

It remains to show that it is possible that $\lfloor|a - b|\rfloor = 71$ for some a and b.

Let $a = 5$ and $b = -66$. Then $x^2+ab+b = (x+11)(x-6)$ and the inequality $x^2+ab+b < 0$ has solutions $\{x : -11 < x < 6\}$. So the set of integer solutions of $x^2 + ab + b < 0$ is really the set of integers in $A \cap B$.

Hence $\max\lfloor|a - b|\rfloor = 71$. □

10. **Answer.** 8

Solution. We consider the polynomial

$$P(t) = t^3 + bt^2 + ct + d.$$

Suppose the root of the equation $P(t) = 0$ are x, y and z. Then

$$-b = x + y + z = 14,$$

$$c = xy + xz + yz = \frac{1}{2}\left((x + y + z)^2 - x^2 - y^2 - z^2\right) = \frac{1}{2}\left(14^2 - 84\right) = 56$$

and

$$x^3 + y^3 + z^3 + 3d = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz).$$

Solving for b, c and d, we get $b = -14$, $c = 30$ and $d = -64$. Finally, since $t^3 - 14t^2 + 30t - 64 = 0$ implies $t = 2$ or $t = 4$ or $t = 8$, we conclude that $\max\{\alpha, \beta, \gamma\} = 8$. □

11. **Answer.** 38

Solution. Let n be an even positive integer. Then each of the following expresses n as the sum of two odd integers: $n = (n - 15) + 15, (n - 25) + 25$ or $(n - 35) + 35$. Note that at least one of $n - 15, n - 25, n - 35$ is divisible by 3, hence n can be expressed as the sum of two composite odd numbers if $n > 38$. Indeed, it can be verified that 38 cannot be expressed as the sum of two composite odd positive integers. □

12. **Answer.** 1936

Solution. We first show that $a+b$ must be a perfect square. The equation $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$ is equivalent to $\frac{a-c}{c}=\frac{c}{b-c}$. Write $\frac{a-c}{c}=\frac{c}{b-c}=\frac{p}{q}$, where $\gcd(p,q)=1$. From $\frac{a-c}{c}=\frac{p}{q}$, we have $\frac{a}{p+q}=\frac{c}{q}$. Since $\gcd(p,q)=1$, we must have q divides c. Similarly from $\frac{b-c}{c}=\frac{q}{p}$, we have $\frac{b}{p+q}=\frac{c}{p}$. Since $\gcd(p,q)=1$, we must have p divides c. Thus $\gcd(p,q)=1$ implies pq divides c. Therefore $\frac{a}{p(p+q)}=\frac{b}{q(p+q)}=\frac{c}{pq}$ is an integer r. Then r divides a, b and c, so that $r=1$ since $\gcd(a,b,c)=1$. Consequently, $a+b=p(p+q)+q(p+q)=(p+q)^2$.

Next the largest square less than or equal to 2011 is $44^2=1936$. As $1936=1892+44$, and $\frac{1}{1892}+\frac{1}{44}=\frac{1}{43}$, where $\gcd(1892,44,43)=1$, we have $a=1892$, $b=44$ and $c=43$ give the largest value of $a+b$. These values of a,b,c can obtained from the identity $\frac{1}{m^2-m}+\frac{1}{m}=\frac{1}{m-1}$. □

13. **Answer.** 10

Solution. Suppose $9[m]<3[n]$. Note that $9[m]=3^p$ and $3[n]=3^q$ for some integers p and q. Thus, $q\geq p+1$. In particular,

$$2(9[m])<3(9[m])=3^{p+1}\leq 3^q=3[n].$$

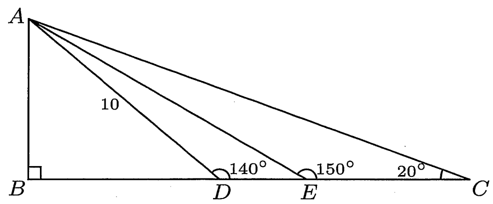
Then we have

$$9[m+1]=(3^2)^{9[m]}=3^{2(9[m])}<3^{3[n]}=3[n+1].$$

Thus, $9[m]<3[n]$ implies $9[m+1]<3[n+1]$. It is clear that $9[2]=81=3^4<3[3]$. Continuing this way, $9[9]<3[10]$. It is also clear that $9[9]>3[9]$, hence the minimum value of n is 10. □

14. **Answer.** 50

Direct calculation gives $\angle DAC=20^\circ$ and $\angle BAD=50^\circ$. Thus $AD=CD=10$. Also $BD=10\sin 50^\circ$. By sine rule applied to the triangle AEC, we have $\frac{CE}{\sin 10^\circ}=\frac{AC}{\sin 150^\circ}=\frac{2\times 10\cos 20^\circ}{\sin 150}=40\cos 20^\circ$. (Note that $AD=DC$.)



Therefore, $BD\cdot CE=400\cos 20^\circ\sin 10^\circ\sin 50^\circ$.

Direct calculation shows that $\cos 20^\circ\sin 10^\circ\sin 50^\circ=\frac{1}{8}$ so that $BD\cdot CE=50$. □

15. **Answer.** 34220

Solution. Note that the condition $a_i\leq a_{i+1}-(i+2)$ for $i=1,2$ is equivalent to that

$$a_1+3\leq a_2,\quad a_2+4\leq a_3.$$

Let A be the set of all 3-element subsets $\{a_1,a_2,a_3\}$ of S such that $a_1+3\leq a_2$ and $a_2+4\leq a_3$.

Let B be the set of all 3-element subsets $\{b_1,b_2,b_3\}$ of the set $\{1,2,\cdots,60\}$.

We shall show that $|A| = |B| = \binom{60}{3} = 34220$ by showing that the mapping ϕ below is a bijection from A to B:

$$\phi : \{a_1, a_2, a_3\} \longrightarrow \{a_1, a_2 - 2, a_3 - 5\}.$$

First, since $\{a_1, a_2, a_3\} \in A$, we have $a_1+3 \le a_2$ and $a_2+4 \le a_3$, and so $a_1 < a_2-2 < a_3-5$, implying that $\{a_1, a_2 - 2, a_3 - 5\} \in B$.

It is clear that ϕ is injective.

It is also surjective, as for any $\{b_1, b_2, b_3\} \in B$ with $b_1 < b_2 < b_3$, we have $\{b_1, b_2+2, b_3+5\} \in A$ and

$$\phi : \{b_1, b_2 + 2, b_3 + 5\} \longrightarrow \{b_1, b_2, b_3\}.$$

Hence ϕ is a bijection and $|A| = |B| = 34220$. □

16. **Answer.** 32

Solution. It is clear that $8(\cos 40°)^3 - 6\cos 40° + 1 = 0$, since $\cos 3A = 4\cos^3 A - 3\cos A$. Observe that

$$\begin{aligned}
& \frac{3}{\sin^2 20°} - \frac{1}{\cos^2 20°} + 64\sin^2 20° \\
= & \frac{6}{1-\cos 40°} - \frac{2}{1+\cos 40°} + 32(1-\cos 40°) \\
= & \frac{8\cos 40° + 4}{1-(\cos 40°)^2} + 32 - 32\cos 40° \\
= & \frac{8\cos 40° + 4 - 32\cos 40° + 32(\cos 40°)^3}{1-(\cos 40°)^2} + 32 \\
= & 4 \times \frac{1 - 6\cos 40° + 8(\cos 40°)^3}{1-(\cos 40°)^2} + 32 \\
= & 32,
\end{aligned}$$

where the last step follows from $8(\cos 40°)^3 - 6\cos 40° + 1 = 0$. □

17. **Answer.** 6029

Solution. Given the original equation

$$f(x^2 + x) + 2f(x^2 - 3x + 2) = 9x^2 - 15x,$$

we replace x by $1 - x$ and obtain

$$f(x^2 - 3x + 2) + 2f(x^2 + x) = 9(1-x)^2 - 15(1-x) = 9x^2 - 3x - 6.$$

Eliminating $f(x^2 - 3x + 2)$ from the two equations, we obtain

$$3f(x^2 + x) = 9x^2 + 9x - 12,$$

thereby

$$f(x^2 + x) = 3x^2 + 3x - 4 = 3(x^2 + x) - 4,$$

hence $f(2011) = 3(2011) - 4 = 6029$.

□

18. **Answer.** 2112

Solution. We denote the numbers of regions divided by n circles by $P(n)$. We have $P(1) = 2,\ P(2) = 4,\ P(3) = 8,\ P(4) = 14, \ldots$ and from this we notice that

$$\begin{aligned} P(1) &= 2, \\ P(2) &= P(1) + 2, \\ P(3) &= P(2) + 4, \\ P(4) &= P(3) + 6, \\ \ldots & \quad \ldots \\ P(n) &= P(n-1) + 2(n-1). \end{aligned}$$

Summing these equations, we obtain

$$P(n) = 2 + 2 + 4 + \ldots + 2(n-1) = 2 + n(n-1).$$

This formula can be shown by induction on n to hold true.
Base case: $n = 1$ is obvious.
Inductive step: Assume that the formula holds for $n = k \geq 1$, i.e., $P(k) = 2 + k(k-1)$. Consider $k+1$ circles, the $(k+1)$-th circle intersects k other circles at $2k$ points (for each one, it cuts twice), which means that this circle is divided into $2k$ arcs, each of which divides the region it passes into two sub-regions. Therefore, we have in addition $2k$ regions, and so

$$P(k+1) = P(k) + 2k = 2 + k(k-1) + 2k = 2 + k(k+1).$$

The proof by induction is thus complete.
Using this result, put $n = 2011$, the number of regions $N = 2 + 2011 \cdot (2011 - 1) = 4042112$. So, the last 4 digits are 2112. □

19. **Answer.** 6034

Solution. Let n be a positive integer.
If $n \leq x < n + \frac{1}{3}$, then $2n \leq 2x < 2n + \frac{2}{3}$ and $3n \leq 3x < 3n + 1$, giving

$$N = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor = n + 2n + 3n = 6n.$$

If $n + \frac{1}{3} \leq x < n + \frac{1}{2}$, then $2n + \frac{2}{3} \leq 2x < 2n + 1$ and $3n + 1 \leq 3x < 3n + \frac{3}{2}$, giving

$$N = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor = n + 2n + 3n + 1 = 6n + 1.$$

If $n + \frac{1}{2} \leq x < n + \frac{2}{3}$, then $2n + 1 \leq 2x < 2n + \frac{4}{3}$ and $3n + \frac{3}{2} \leq 3x < 3n + \frac{4}{3}$, giving

$$N = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor = n + 2n + 1 + 3n + 1 = 6n + 2.$$

If $n + \frac{2}{3} \leq x < n + 1$, then $2n + \frac{4}{3} \leq 2x < 2n + 2$ and $3n + 2 \leq 3x < 3n + 3$, giving

$$N = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor = n + 2n + 1 + 3n + 2 = 6n + 3.$$

Thus, "invisible" numbers must be of the form $6n + 4$ and $6n + 5$. The 2011^{th} "invisible" integer is $4 + 6 \times \dfrac{2011 - 1}{2} = 6034$. □

20. **Answer.** 95004

Solution. We shall prove that for any positive integer a, if $f(a)$ denotes the sum of all nonnegative integer solutions to $\lfloor \frac{n}{a} \rfloor = \lfloor \frac{n}{a+1} \rfloor$, then

$$f(a) = \frac{1}{6}a(a^2-1)(a+2).$$

Thus $f(27) = 95004$.

Let n be a solution to $\lfloor \frac{n}{a} \rfloor = \lfloor \frac{n}{a+1} \rfloor$. Write $n = aq + r$, where $0 \le r < a$. Thus $\lfloor \frac{n}{a} \rfloor = q$. Also $n = (a+1)q + r - q$. Since $\lfloor \frac{n}{a+1} \rfloor = q$, we have $0 \le r - q$, that is, $q \le r < a$. Therefore for each $q = 0, 1, \ldots, a-1$, r can be anyone of the values $q, q+1, \ldots, a-1$. Thus

$$\begin{aligned}
A &= \sum_{q=0}^{a-1}\sum_{r=q}^{a-1}(qa+r) \\
&= \sum_{q=0}^{a-1}(a-q)qa + \sum_{q=0}^{a-1}\sum_{r=q}^{a-1} r \\
&= a^2\sum_{q=0}^{a-1} q - a\sum_{q=0}^{a-1} q^2 + \sum_{r=0}^{a-1}\sum_{q=0}^{r} r \\
&= a^2\sum_{q=0}^{a-1} q - a\sum_{q=0}^{a-1} q^2 + \sum_{r=0}^{a-1} r(r+1) \\
&= a^2\sum_{q=0}^{a-1} q - a\sum_{q=0}^{a-1} q^2 + \sum_{r=0}^{a-1} r^2 + \sum_{r=0}^{a-1} r \\
&= (a^2+1)\cdot\frac{1}{2}a(a-1) + (1-a)\cdot\frac{1}{6}a(2a-1)(a-1) \\
&= \frac{1}{6}a(a^2-1)(a+2).
\end{aligned}$$

□

21. **Answer.** 48

By using factor formulae and double angle formulae:

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{12}{7},$$

and

$$\sin A \sin B \sin C = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{12}{25}.$$

Solving these equations, we obtain

$$\begin{aligned}
\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &= 0.1 \\
\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} &= 0.6
\end{aligned}$$

Furthermore,

$$\sin\frac{C}{2} = \cos\left(\frac{A+B}{2}\right) = \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2},$$

multiplying both sides by $\sin\frac{C}{2}\cos\frac{C}{2}$, we get

$$\sin^2\frac{C}{2}\cos\frac{C}{2} = 0.6\sin\frac{C}{2} - 0.1\cos\frac{C}{2}.$$

or equivalently,

$$(1-t^2)t = 0.6\sqrt{1-t^2} - 0.1t \iff 11t - 10t^3 = 6\sqrt{1-t^2},$$

where $t = \cos\frac{C}{2}$. This equation solves for $t = \sqrt{\frac{1}{2}},\ \sqrt{\frac{4}{5}},\ \sqrt{\frac{3}{10}}$, and so the corresponding values of $\sin C$ are

$$1,\ 0.8,\ 0.6$$

and hence $100s_1s_2s_3 = 100\cdot 1\cdot 0.8\cdot 0.6 = 48$. □

22. **Answer.** 8

Solution. We first prove that if $x \geq 8$, then $z = 2$. To this end, we observe that the left hand side of the equation $1!+2!+3!+\ldots+x!$ is divisible by 3, and hence $3 \mid y^z$. Since 3 is a prime, $3 \mid y$. So, $3^z \mid y^z$ by elementary properties of divisibility.

On the other hand, when $x = 8$,

$$1!+2!+\ldots+8! = 46233$$

is divisible by 3^2 but not by 3^3. Now, note that if $n \geq 9$, then we have $3^3 \mid n!$. So, when $x \geq 8$, the left hand side is divisible by 3^2 but not by 3^3. This means that $z = 2$.

We now prove further that when $x \geq 8$, then the given equation has no solutions. To prove this, we observe that $x \geq 8$ implies that

$$1!+2!+3!+4!+\underbrace{5!+\ldots x!}_{\text{divisible by 5}} \equiv 3 \pmod 5.$$

Since we have deduced that $z = 2$, we only have $y^2 \equiv 0,\ 1,\ -1 \pmod 5$. This mismatch now completes the argument that there are no solutions to the equation when $x \geq 8$.

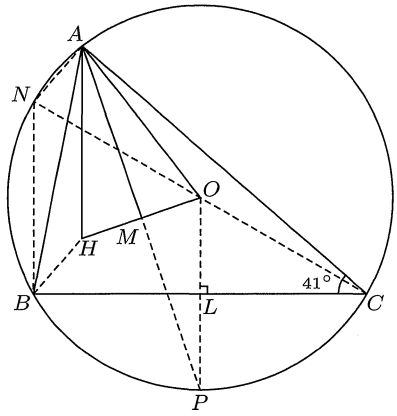
So the search narrows down to $x < 8$. By exhaustion, it is easy to find that there is only one solution:

$$x = y = 3,\ z = 2.$$

Thus, the sum of this only combination must be the largest and is equal to $3+3+2 = 8$.
□

23. **Answer.** 38

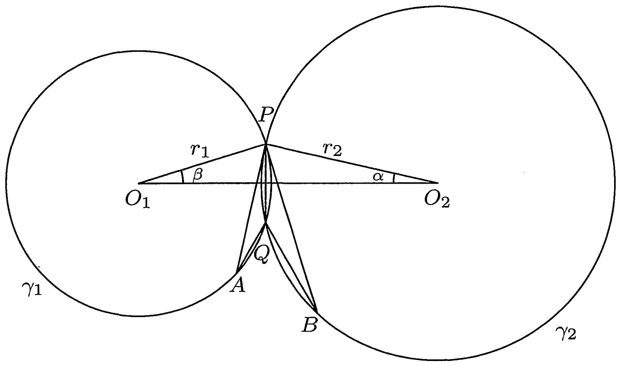
Let P be the midpoint of the arc BC not containing A on the circumcircle of the triangle ABC. Then OP is the perpendicular bisector of BC. Since AM bisects $\angle A$, the points A, M, P are collinear. As both AH and OP are perpendicular to BC, they are parallel. Thus $\angle HAM = \angle OPM = \angle OAM$. Also $\angle HMA = \angle OMP$. Since $HM = OM$, we have the triangles AHM and POM are congruent. Therefore $AH = PO = AO$.



Let L be the midpoint of BC. It is a known fact that $AH = 2OL$. To see this, extend CO meeting the circumcircle of the triangle ABC at the point N. Then $ANBH$ is a parallelogram. Thus $AH = NB = 2OL$. Therefore in the right-angled triangle OLC, $OC = OA = AH = 2OL$. This implies $\angle OCL = 30°$. Since the triangle ABC is acute, the circumcentre O lies inside the triangle. In fact $\angle A = 60°$ and $\angle B = 79°$. Then $\angle OAC = \angle OCA = 41° - 30° = 11°$. Consequently, $\angle HAO = 2\angle OAM = 2 \times (30° - 11°) = 38°$. □

24. **Answer.** 30

Let $PO_1 = r_1$ and $PO_2 = r_2$. First note that O_1O_2 intersects PQ at the midpoint H (not shown in the figure) of PQ perpendicularly. Next observe that $\angle APQ = \angle PBQ = \angle PO_2O_1$, and $\angle BPQ = \angle PAQ = \angle PO_1O_2$. Therefore $\angle APB = \angle APQ + \angle BPQ = \angle PO_2O_1 + \angle PO_1O_2$.



Let $\angle PO_2O_1 = \alpha$ and $\angle PO_1O_2 = \beta$. Then $\sin\alpha = \frac{PQ}{2r_2}, \cos\alpha = \frac{O_2H}{r_2}$ and $\sin\beta = \frac{PQ}{2r_1}, \cos\beta = \frac{O_1H}{r_1}$. Thus $\sin\angle APB = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{PQ}{2r_2}\cdot\frac{O_1H}{r_1} + \frac{O_2H}{r_2}\cdot\frac{PQ}{2r_1} = \frac{PQ\cdot(O_1H+O_2H)}{2r_1r_2} = \frac{PQ\cdot O_1O_2}{2r_1r_2} = \frac{1}{2}$. Since $\angle APB$ is acute, it is equal to 30°. □

25. **Answer.** 2

Solution. Let

$$a_n = \sum_{i=0}^{n} \binom{n}{i}^{-1}.$$

Assume that $n \geq 3$. It is clear that

$$a_n = 2 + \sum_{i=1}^{n-1} \binom{n}{i}^{-1} > 2.$$

Also note that

$$a_n = 2 + 2/n + \sum_{i=2}^{n-2} \binom{n}{i}^{-1}.$$

Since $\binom{n}{i} \geq \binom{n}{2}$ for all i with $2 \leq i \leq n-2$,

$$a_n \leq 2 + 2/n + (n-3)\binom{n}{2}^{-1} \leq 2 + 2/n + 2/n = 2 + 4/n.$$

So we have show that for all $n \geq 3$,

$$2 < a_n \leq 2 + 4/n.$$

Thus

$$\lim_{n\to\infty} a_n = 2.$$

□