Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2022
(Open Section, Round 1)

Thursday, 2 June 2022 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

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In this paper, let $\mathbb{R}$ denote the set of all real numbers, and $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$.

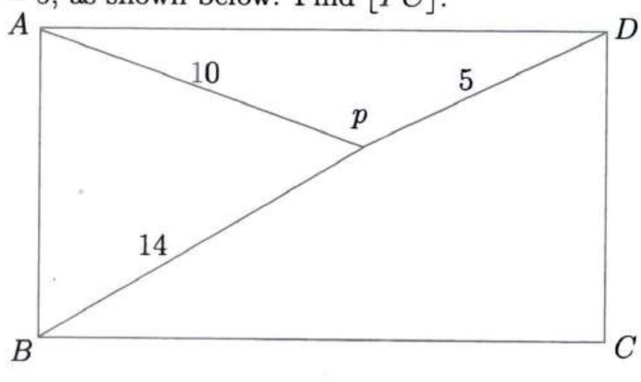
1. If $S = \displaystyle\sum_{k=-2021}^{2021} \frac{1}{10^k + 1}$, find $\lfloor 2S \rfloor$.

2. All the positive integers $1, 2, 3, 4, \cdots$, are grouped in the following way: $G_1 = \{1, 2\}$, $G_2 = \{3, 4, 5, 6\}$, $G_3 = \{7, 8, 9, 10, 11, 12, 13, 14\}$, that is, the set G_n contains the next 2^n positive integers listed in ascending order after the set G_{n-1}, $n > 1$. If S is the sum of all the positive integers from G_1 to G_8, find $\left\lfloor \dfrac{S}{100} \right\rfloor$.

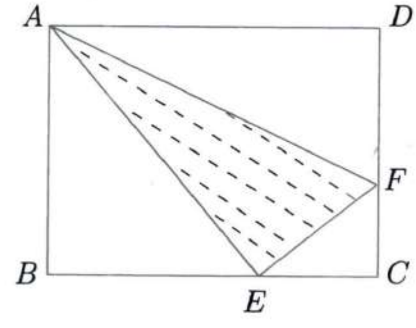
3. A sequence of one hundred positive integers $x_1, x_2, x_3, \cdots, x_{100}$ are such that
$$(x_1)^2 + (2x_2)^2 + (3x_3)^2 + (4x_4)^2 + \cdots + (100x_{100})^2 = 338350.$$
Find the largest possible value of $x_1 + x_2 + x_3 + \cdots + x_{100}$.

4. Let a and b be two real numbers satisfying $a < b$, and such that for each real number m satisfying $a < m < b$, the circle $x^2 + (y - m)^2 = 25$ meets the parabola $4y = x^2$ at four distinct points in the Cartesian plane. Let S be the maximum possible value of $b - a$. Find $\lfloor 4S \rfloor$.

5. Let P be a point within a rectangle $ABCD$ such that $PA = 10$, $PB = 14$ and $PD = 5$, as shown below. Find $\lfloor PC \rfloor$.



6. In the diagram below, the rectangle $ABCD$ has area 180 and both triangles ABE and ADF have areas 60. Find the area of triangle AEF.

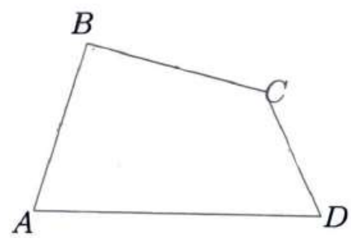


7. A tetrahedron in $\mathbb{R}^3$ has one vertex at the origin O and the other vertices at the points $A(6,0,0)$, $B(4,2,4)$ and $C(3,2,6)$. If x is the height of the tetrahedron from O to the plane ABC, find $\lfloor 5x^2 \rfloor$.

8. Let x and y be real numbers such that $(x-2)^2 + (y-3)^2 = 4$. If S is the largest possible value of $x^2 + y^2$, find $\lfloor (S-17)^2 \rfloor$.

9. Let S be the maximum value of $w^3 - 3w$ subject to the condition that $w^4 + 9 \leq 10w^2$. Find $\lfloor S \rfloor$.

10. In the quadrilateral $ABCD$ below, it is given that $AB = BC = CD$ and $\angle ABC = 80°$ and $\angle BCD = 160°$. Suppose $\angle ADC = x°$. Find the value of x.



11. Let a, b, c be integers with $ab + c = 49$ and $a + bc = 50$. Find the largest possible value of abc.

12. Find the largest possible value of $|a| + |b|$, where a and b are coprime integers (i.e., a and b are integers which have no common factors larger than 1) such that $\frac{a}{b}$ is a solution of the equation below:

$$\sqrt{4x + 5 - 4\sqrt{x+1}} + \sqrt{x + 2 - 2\sqrt{x+1}} = 1.$$

13. Let S be the set of real solutions (x, y, z) of the following system of equations:

$$\begin{cases} \dfrac{4x^2}{1+4x^2} = y, \\ \dfrac{4y^2}{1+4y^2} = z, \\ \dfrac{4z^2}{1+4z^2} = x. \end{cases}$$

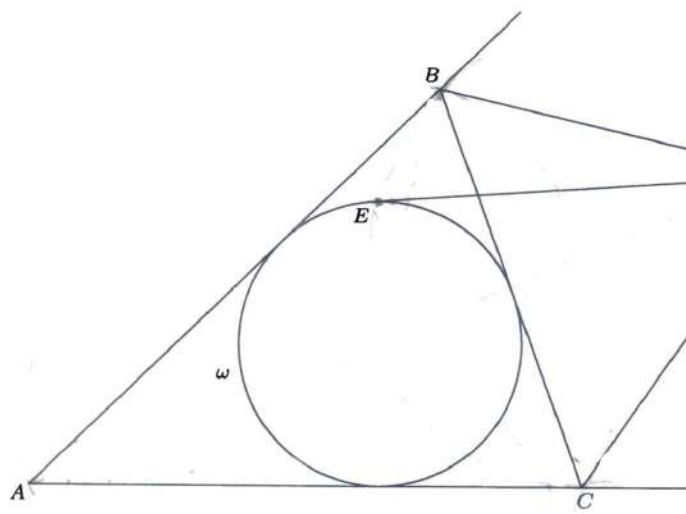
For each $(x, y, z) \in S$, define $m(x, y, z) = 2000(|x| + |y| + |z|)$. Determine the maximum value of $m(x, y, z)$ over all $(x, y, z) \in S$.

14. Assume that t is a positive solution to the equation

$$t = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + t}}}}.$$

Determine the value of $t^4 - t^3 - t + 10$.

15. In the triangle ABC shown in the diagram below, the external angle bisectors of $\angle B$ and $\angle C$ meet at the point D. The tangent from D to the incircle ω of the triangle ABC touches ω at E, where E and B are on the same side of the line AD. Suppose $\angle BEC = 112°$. Find the size of $\angle A$ in degrees.



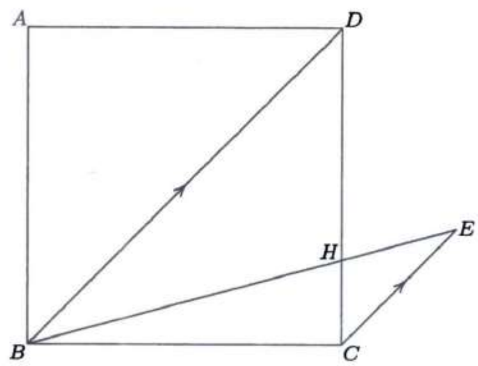
16. Find the largest integer n such that $n^2 + 5n - 9486 = 10s(n)$, where $s(n)$ is the product of all digits of n in the decimal representation of n.

(For example, $s(481) = 4 \times 8 \times 1 = 32$.)

17. Find the number of integer solutions to the equation $19x + 93y = 4xy$.

18. Find the number of integer solutions to the equation $x_1 + x_2 - x_3 = 20$ with $x_1 \geq x_2 \geq x_3 \geq 0$.

19. In the diagram below, E is a point outside a square $ABCD$ such that CE is parallel to BD, $BE = BD$, and BE intersects CD at H. Given $BE = \sqrt{6} + \sqrt{2}$, find the length of DH.



20. The diagram below shows the region $R = \{(x, y) \in \mathbb{R}^2 \mid y \geq \frac{1}{2}x^2\}$ on the xy-plane bounded by the parabola $y = \frac{1}{2}x^2$. Let C_1 be the largest circle lying inside R with its lowest point at the origin. Let C_2 be the largest circle lying inside R and resting on top of C_1. Find the sum of radii of C_1 and C_2.

21. Find the smallest positive integer x such that $3x^2 + x = 4y^2 + y$ for some positive integer y.

22. A group of students participate in some sports activities among 6 different types of sports. It is known that for each sports activity there are exactly 100 students in the group participating in it; and the union of all the sports activities participated by any two students is NOT the entire set of 6 sports activities. Determine the minimum number of students in the group.

23. Let p and q be positive prime integers such that $p^3 - 5p^2 - 18p = q^9 - 7q$. Determine the smallest value of p.

24. Given that a, b, c are positive real numbers such that $a+b+c=9$, find the maximum value of $a^2b^3c^4$.

25. Let $\mathbb{R}^+$ be the set of all positive real numbers. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying

$$xyf(x)\left(f(y) - f(yf(x))\right) = 1$$

for all $x, y \in \mathbb{R}^+$. Find $f(\frac{1}{2022})$.