Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2013
(Open Section, First round)

Wednesday, 5 June 2013 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

1. The sum

$$\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+\frac{1}{3\times 4\times 5}+\cdots+\frac{1}{100\times 101\times 102}$$

can be expressed as $\frac{a}{b}$, a fraction in its simplest form. Find $a+b$.

2. Determine the maximum value of $\frac{1+\cos x}{\sin x+\cos x+2}$, where x ranges over all real numbers.

3. Let $\tan\alpha$ and $\tan\beta$ be two solutions of the equation $x^2-3x-3=0$. Find the value of

$$\left|\sin^2(\alpha+\beta)-3\sin(\alpha+\beta)\cos(\alpha+\beta)-3\cos^2(\alpha+\beta)\right|.$$

(Note: $|x|$ denotes the absolute value of x.)

4. Suppose that $a_1, a_2, a_3, a_4, \cdots$ is an arithmetic progression with $a_1>0$ and $3a_8=5a_{13}$. Let $S_n=a_1+a_2+\cdots+a_n$ for all integers $n\geq 1$. Find the integer n such that S_n has the maximum value.

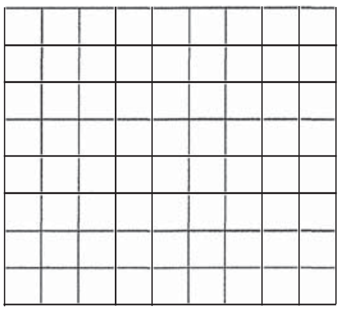
5. If $\mathrm{g}(x)=\tan\frac{x}{2}$ for $0<x<\pi$ and $\mathrm{f}(\mathrm{g}(x))=\sin 2x$, find the value of k such that $k\mathrm{f}(\frac{\sqrt{2}}{2})=36\sqrt{2}$.

6. Let $\mathrm{g}(x)$ be a strictly increasing function defined for all $x\geq 0$. It is known that the range of t satisfying

$$\mathrm{g}(2t^2+t+5)<\mathrm{g}(t^2-3t+2)$$

is $b<t<a$. Find $a-b$.

7. The figure below shows an 8×9 rectangular board.



How many squares are there in the above rectangular board?

8. Let a, b, c be positive real numbers such that $a+b+c=2013$. Find the maximum value of $\sqrt{3a+12}+\sqrt{3b+12}+\sqrt{3c+12}$.

9. Let $A=\cos^2 10^\circ+\cos^2 50^\circ-\sin 40^\circ\sin 80^\circ$. Determine the value of $100A$.

10. Assume that $a_i \in \{1, -1\}$ for all $i = 1, 2, \cdots, 2013$. Find the least positive number of the following expression

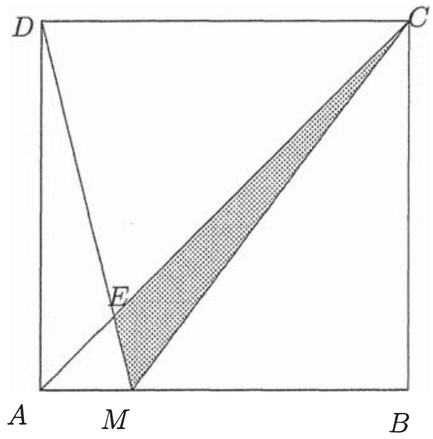
$$\sum_{1 \leq i < j \leq 2013} a_i a_j.$$

11. Let f be a function defined on non-zero real numbers such that

$$\frac{27\mathrm{f}(-x)}{x} - x^2 \mathrm{f}\left(\frac{1}{x}\right) = -2x^2,$$

for all $x \neq 0$. Find f(3).

12. In the figure below, $ABCD$ is a square with $AB = 20$ cm (not drawn to scale). Assume that M is a point such that the area of the shaded region is 40 cm^2. Find AM in centimetres.



13. In the triangle ABC, a circle passes through the point A, the midpoint E of AC, the midpoint F of AB and is tangent to the side BC at D. Suppose

$$\frac{AB}{AC} + \frac{AC}{AB} = 4.$$

Determine the size of $\angle EDF$ in degrees.

14. Let $a_1, a_2, a_3, \cdots$ be a sequence of real numbers in a geometric progression. Let $S_n = a_1 + a_2 + \cdots + a_n$ for all integers $n \geq 1$. Assume that $S_{10} = 10$ and $S_{30} = 70$. Find the value of S_{40}.

15. Find the number of three-digit numbers which are multiples of 3 and are formed by the digits 0,1,2,3,4,5,6,7 without repetition.

16. All the positive integers which are co-prime to 2012 are grouped in an increasing order in such a way that the n^{th} group has $2n-1$ numbers. So, the first three groups in this grouping are $(1), (3,5,7), (9,11,13,15,17)$. It is known that 2013 belongs to the k^{th} group. Find the value of k.

(Note: Two integers are said to be co-prime if their greatest common divisor is 1.)

17. The numbers $1, 2, 3, \cdots, 7$ are randomly divided into two non-empty subsets. The probability that the sum of the numbers in the two subsets being equal is $\frac{p}{q}$ expressed in the lowest term. Find $p+q$.

18. Find the number of real roots of the equation $\log_{10}^2 x - \lfloor \log_{10} x \rfloor - 2 = 0$.

(Note: $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.)

19. In the triangle ABC, $AB = AC$, $\angle A = 90°$, D is the midpoint of BC, E is the midpoint of AC and F is a point on AB such that BE intersects CF at P and B, D, P, F lie on a circle. Let AD intersect CP at H. Given $AP = \sqrt{5} + 2$, find the length of PH.

20. Find the total number of positive integers n not more than 2013 such that $n^4 + 5n^2 + 9$ is divisible by 5.

21. In a circle ω centred at O, AA' and BB' are diameters perpendicular to each other such that the points A, B, A', B' are arranged in an anticlockwise sense in this order. Let P be a point on the minor arc $A'B'$ such that AP intersects BB' at D and BP intersects AA' at C. Suppose the area of the quadrilateral $ABCD$ is 100. Find the radius of ω.

22. A sequence $a_1, a_2, a_3, a_4, \cdots$, with $a_1 = \frac{1}{2}$, is defined by

$$a_n = 2a_n a_{n+1} + 3a_{n+1}$$

for all $n = 1, 2, 3, \cdots$. If $b_n = 1 + \dfrac{1}{a_n}$ for all $n = 1, 2, 3, \cdots$, find the largest integer m such that

$$\sum_{k=1}^{n} \frac{1}{\log_3 b_k} > \frac{m}{24}$$

for all positive integer $n \geq 2$.

23. Find the largest real number p such that all three roots of the equation below are positive integers:

$$5x^3 - 5(p+1)x^2 + (71p-1)x + 1 = 66p.$$

24. Let a, b, c, d be 4 distinct nonzero integers such that $a + b + c + d = 0$ and the number $M = (bc - ad)(ac - bd)(ab - cd)$ lies strictly between 96100 and 98000. Determine the value of M.

25. In the triangle ABC, $AB = 585$, $BC = 520$, $CA = 455$. Let P, Q be points on the side BC, and $R \neq A$ the intersection of the line AQ with the circumcircle ω of the triangle ABC. Suppose PR is parallel to AC and the circumcircle of the triangle PQR is tangent to ω at R. Find PQ.