Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2013
(Open Section, First round Solution)

1. Answer: 12877

   Solution. Let S be the required sum. By using method of difference,

$$\begin{aligned} S &= \frac{1}{2}\left(\frac{1}{1\times 2}-\frac{1}{2\times 3}+\frac{1}{2\times 3}-\frac{1}{3\times 4}+\cdots+\frac{1}{100\times 101}-\frac{1}{101\times 102}\right) \\ &= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{10302}\right) \\ &= \frac{2575}{10302}, \end{aligned}$$

   Hence $a+b=12877$. □

2. Answer: 1

   Solution. Let $y=\dfrac{1+\cos x}{\sin x+\cos x+2}$. When $\cos x+1=0$, $y=0$. Otherwise,

$$y=\frac{1}{1+\frac{1+\sin x}{1+\cos x}}.$$

   Let $u=\dfrac{1+\sin x}{1+\cos x}$. It is clear that $u\geq 0$, and so $y\leq 1$ where the equality holds when $u=0$.

   Thus the maximum value of y is 1 when $\sin x=-1$. □

3. Answer: 3

   Solution. We have $\tan\alpha+\tan\beta=3$ and $\tan\alpha\tan\beta=-3$. Hence

$$\tan(\alpha+\beta)=\frac{3}{1-(-3)}=\frac{3}{4}.$$

   Hence

$$\begin{aligned} &\left|\sin^2(\alpha+\beta)-3\sin(\alpha+\beta)\cos(\alpha+\beta)-3\cos^2(\alpha+\beta)\right| \\ =\ & \cos^2(\alpha+\beta)\left[\left(\frac{3}{4}\right)^2-3\left(\frac{3}{4}\right)-3\right] \\ =\ & \cos^2(\alpha+\beta)\times\left(-\frac{75}{16}\right), \end{aligned}$$

   and since

$$\tan^2(\alpha+\beta)=\frac{1-\cos^2(\alpha+\beta)}{\cos^2(\alpha+\beta)}=\frac{1}{\cos^2(\alpha+\beta)}-1=\frac{9}{16},$$

   we have $\cos^2(\alpha+\beta)=\dfrac{16}{25}$. Thus, we have

$$\left|\sin^2(\alpha+\beta)-3\sin(\alpha+\beta)\cos(\alpha+\beta)-3\cos^2(\alpha+\beta)\right|=|-3|=3.$$

   □

4. Answer: 20

Solution. Let $a_n = a_1 + (n-1)d$. As

$$3a_8 = 5a_{13},$$

we have $3(a_1 + 7d) = 5(a_1 + 12d)$, and so $2a_1 + 39d = 0$. So $d < 0$ and

$$a_{20} + a_{21} = a_1 + 19d + a_1 + 20d = 0.$$

So $a_{20} > 0$ but $a_{21} < 0$, as $a_{21} = a_{20} + d$ and $a_{20} + a_{21} = 0$. Thus $a_1, a_2, a_3, a_4, \cdots$ is an decreasing sequence and

$$a_1 > a_2 > \cdots > a_{20} > 0 > a_{21} > \cdots.$$

Hence S_n has the maximum value when $n = 20$. □

5. Answer: 81

Solution. Note that $\mathrm{f}(\mathrm{g}(x)) = \sin 2x = 2\sin x \cos x = \dfrac{4\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \cdot \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$. Hence

$$\mathrm{f}(\frac{\sqrt{2}}{2}) = \frac{2\sqrt{2}}{1+\frac{1}{2}} \cdot \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{4\sqrt{2}}{9}.$$

If $k\mathrm{f}(\dfrac{\sqrt{2}}{2}) = 36\sqrt{2}$, then $k = 81$. □

6. Answer: 2

Solution. Note that $2t^2+t+5 = 2\left(t+\dfrac{1}{4}\right)^2 + \dfrac{39}{8} > 0$. Hence $\mathrm{g}(2t^2+t+5) < \mathrm{g}(t^2-3x+2)$ is true if and only if

$$2t^2 + t + 5 < t^2 - 3t + 2,$$

which is equivalent to $(t+3)(t+1) < 0$. Hence the range of t satisfying the given inequality is $-3 < t < -1$, which yields $a - b = (-1) - (-3) = 2$. □

7. Answer: 240

Solution. By counting the number of squares of different types, we obtain

$$1\times 2 + 2\times 3 + 3\times 4 + 4\times 5 + 5\times 6 + 6\times 7 + 7\times 8 + 8\times 9 = \frac{8\times 9\times 10}{3} = 240.$$

□

8. Answer: 135

Solution. Note that

$$(\sqrt{3a+12} + \sqrt{3b+12} + \sqrt{3c+12})^2 \le 3(3a+12+3b+12+3c+12)$$
$$= 9(a+b+c+12) = 9(2013+12) = 9\times 2025,$$

where the equality holds if $3a + 12 = 3b + 12 = 3c + 12$, i.e., $a = b = c = 671$. Thus the answer is $3 \times 45 = 135$.

□

9. Answer: 75

Solution. Let
$$B = \sin^2 10^\circ + \sin^2 50^\circ - \cos 40^\circ \cos 80^\circ.$$
Then
$$A + B = 2 - \cos 40^\circ$$
and
$$A - B = \cos 20^\circ + \cos 100^\circ + \cos 120^\circ = 2\cos 60^\circ \cos 40^\circ + \cos 120^\circ$$
$$= \cos 40^\circ - \frac{1}{2}.$$
Thus $2A = \frac{3}{2}$ and $100A = 75$. □

10. Answer: 6

Solution. Note that
$$2 \sum_{1 \le i < j \le 2013} a_i a_j = (a_1 + a_2 + \cdots + a_{2013})^2 - (a_1^2 + a_2^2 + \cdots + a_{2013}^2)$$
$$= (a_1 + a_2 + \cdots + a_{2013})^2 - 2013.$$
By the given condition, $a_1 + a_2 + \cdots + a_{2013}$ is an odd number between -2013 and 2013 inclusive.

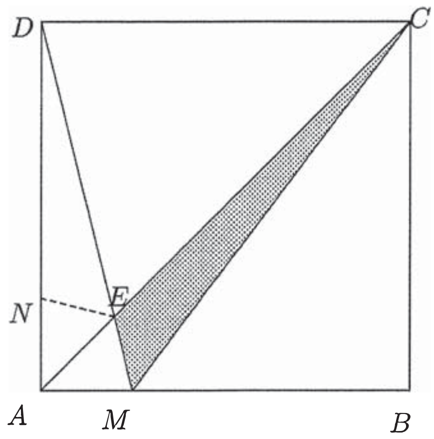
Also note that the minimum positive integer of $x^2 - 2013$ for an integer x is $45^2 - 2013 = 12$ when $x = 45$ or -45. As an illustration, $x = 45$ can be achieved by taking $a_1 = a_2 = a_3 = \cdots = a_{45} = 1$ and the others $a_{46}, a_{47}, \cdots, a_{2013}$ to consist of equal number of 1's and -1's. Thus the least value is $\frac{12}{2} = 6$. □

11. Answer: 2

Solution. Letting $x = -y$, we get
$$-\frac{27\mathrm{f}(y)}{y} - y^2 \mathrm{f}\left(-\frac{1}{y}\right) = -2y^2. \tag{1}$$
Letting $x = \frac{1}{y}$, we get
$$27y\mathrm{f}\left(-\frac{1}{y}\right) - \frac{1}{y^2}\mathrm{f}(y) = -\frac{2}{y^2}. \tag{2}$$
Then $27 \times (1) + y \times (2)$ gives
$$-\frac{729\mathrm{f}(y)}{y} - \frac{\mathrm{f}(y)}{y} = -54y^2 - \frac{2}{y}.$$
Solving for $\mathrm{f}(y)$, we have $\mathrm{f}(y) = \frac{1}{365}(27y^3 + 1)$. Thus $\mathrm{f}(3) = \frac{3^6+1}{365} = 2$. □

12. Answer: 5

Solution. Choose a point N on DA such that $NA = MA = x$.



It is clear that ΔNAE and ΔMAE are congruent by SAS test. Let S be the area of $\triangle NAE$. Then area of $\Delta DNE = \frac{20-x}{x}S$. It is also clear that areas of ΔDAE and ΔCEM are equal to 40cm^2. It follows that

$$\text{Area of } \Delta DAE = \frac{20-x}{x}S + S = \frac{20}{x}S,$$

so that $\frac{20}{x}S = 40\text{cm}^2$, that is, $S = 2x$.

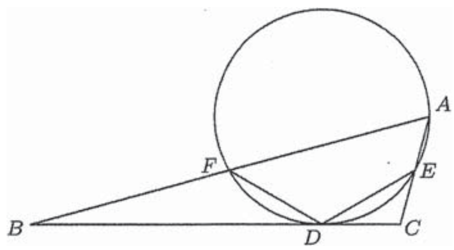
Area of $\Delta DAM = \frac{1}{2} \times x \times 20 = 10x$. On the other hand,

$$\begin{aligned}\text{Area of } \Delta DAM &= \text{Area of} \Delta DAE + \text{Area of} \Delta AEM \\ &= \frac{20}{x}S + S \\ &= \frac{20+x}{x}S = \frac{20+x}{x} \times 2x = 2(20+x).\end{aligned}$$

So $2(20+x) = 10x$, which means that $AM = x = 5$cm. □

13. Answer: 120

Solution.



Let $BC = a, CA = b$ and $AB = c$. Let $BD = a_1$ and $DC = a_2$. Using the power of B with respect to the circle, we have $a_1^2 = c^2/2$. Similarly, $a_2^2 = b^2/2$. Thus $b + c = \sqrt{2}(a_1 + a_2) =$

$\sqrt{2}a$, or $2a^2 = (b+c)^2$. Therefore,

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{2b^2+2c^2-(b+c)^2}{4bc} = \frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}-2\right) = \frac{1}{4}(4-2) = \frac{1}{2}.$$

Therefore, $\angle A = 60°$. Since A, F, D, E are concyclic, $\angle EDF = 120°$.

□

14. Answer: 150

Solution. Let r be the common ratio of this geometric sequence. Thus

$$S_n = a_1(1 + r + r^2 + \cdots + r^{n-1}).$$

Thus

$$10 = a_1(1 + r + \cdots + r^9)$$

and

$$70 = a_1(1 + r + \cdots + r^{29}).$$

As

$$1 + r + \cdots + r^{29} = (1 + r + \cdots + r^9)(1 + r^{10} + r^{20}),$$

we have

$$1 + r^{10} + r^{20} = 7.$$

So r^{10} is either 2 or -3. As $r^{10} > 0$, $r^{10} = 2$.

Hence

$$S_{40} = a_1(1 + r + \cdots + r^{39}) = a_1(1 + r + \cdots + r^9)(1 + r^{10} + r^{20} + r^{30})$$
$$= 10 \times (1 + 2 + 2^2 + 2^3) = 150.$$

□

15. Answer: 106

Solution. Note that an integer is a multiple of 3 if and only if the sum of its digits is a multiple of 3. Also note that the sum of three integers a, b, c is a multiple of 3 if and only if either (i) a, b, c all have the same remainder when divided by 3, or (ii) a, b, c have the distinct remainders when divided by 3. Observe that the remainders of $0, 1, 2, 3, 4, 5, 6, 7$ when divided by 3 are $0, 1, 2, 0, 1, 2, 0, 1$ respectively.

For case (i), the only possible selections such that all the three numbers have the same remainder when divided by 3 are $\{0, 3, 6\}$ and $\{1, 4, 7\}$. With $\{0, 3, 6\}$, we have 4 possible numbers (note that a number does not begin with 0), and with $\{1, 4, 7\}$, there are 6 possible choices.

For case (ii), if the choice of the numbers does not include 0, then there are $2\times3\times2\times3! = 72$; if 0 is included, then there are $3 \times 2 \times 4$ choices.

Hence the total number of possible three-digit numbers is $72 + 24 + 10 = 106$.

□

16. Answer: 32

Solution. Note that $2012 = 2^2 \times 503$, and that 503 is a prime number. There are 1006 multiples of 2 less than or equal to 2012; there are 4 multiples of 503 less than or equal to 2012; there are 2 multiples of 1006 less than or equal to 2012. By the Principle of Inclusion and Exclusion, there are $1006 + 4 - 2 = 1008$ positive integers not more than 2012 which are not co-prime to 2012. Hence there are $2012 - 1008 = 1004$ positive integers less than 2012 which are co-prime with 2012. Thus, 2013 is the 1005^{th} number co-prime with 2012. Note also that the sum of the first n odd numbers equals n^2, and that $31^2 < 1005 < 32^2$, the number 2013 must be in the 326th group. Hence $k = 32$. □

17. Answer: 67

The total number of ways of dividing the seven numbers into two non-empty subsets is $\dfrac{2^7 - 2}{2} = 63$. Note that since $1 + 2 + 3 + \cdots + 7 = 28$, the sum of the numbers in each of the two groups is 14. Note also that the numbers $5, 6, 7$ cannot be in the same group since $5 + 6 + 7 = 18 > 14$. We consider three separate cases:

Case (i): Only 6 and 7 in the same group and 5 in the other group:

$\{2, 3, 4, 5\}, \{1, 6, 7\}$

Case (ii): Only 5 and 6 in the same group and 7 in the other group:

$\{1, 2, 5, 6\}, \{3, 4, 7\}$

$\{3, 5, 6\}, \{1, 2, 4, 7\}$

Case (iii): Only 5 and 7 in the same group and 6 in the other group:

$\{2, 5, 7\}, \{1, 3, 4, 6\}$

Hence there are 4 such possibilities. Thus the required probability is $\dfrac{4}{63}$, yielding that $p + q = 67$. □

18. Answer: 3

Solution. Let $u = \lfloor \log_{10} x \rfloor$ and $r = \log_{10} x - u$. So $0 \leq r < 1$. Thus

$$(u + r)^2 = u + 2.$$

Case 1: $r = 0$.

Then $u^2 = u + 2$ and so $u = 2$ or $u = -1$, corresponding to $x = 10^2 = 100$ and $x = 10^{-1} = 0.1$.

Case 2: $0 < r < 1$.

In this case, $u + 2$ is an integer which is not a complete square and

$$r = \sqrt{u + 2} - u.$$

As $r > 0$, we have $u \leq 2$. But $u + 2$ is not a complete square. So $u \leq 1$. As $u + 2 \geq 0$ and not a complete square, we have $u \geq 0$. Hence $u \in \{0, 1\}$.

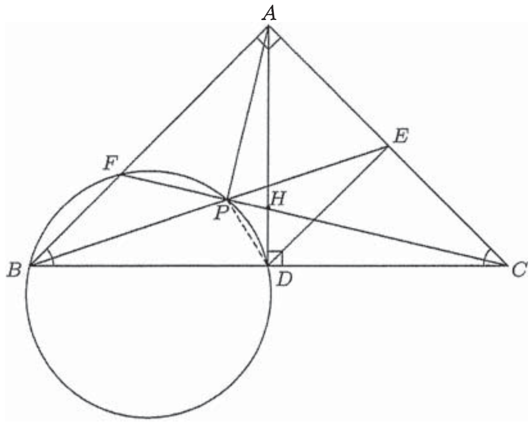
If $u = 0$, then $r = \sqrt{2} - 0 > 1$, not suitable.

If $u = 1$, then $r = \sqrt{3} - 1$. So $\log_{10} x = \sqrt{3}$ and $x = 10^{\sqrt{3}}$.

Hence the answer is 3. □

19. Answer: 1

Solution.



Join PD. Then $\angle DPC = \angle FBD = 45^\circ = \angle DAC$ so that D, P, A, C are concyclic. Thus $\angle APC = \angle ADC = 90^\circ$. It follows that $EA = EP = ED = EC$. Let $\angle PAH = \theta$. Then $\angle PCD = \theta$. Thus $\angle EPC = \angle ECP = 45^\circ - \theta$ so that $\angle AEB = 90^\circ - 2\theta$. That is $\angle ABE = 2\theta$. Thus $\tan 2\theta = AE/AB = 1/2$. From this, we get $\tan\theta = \sqrt{5} - 2$. Therefore, $PH = AP\tan\theta = (\sqrt{5}+2)(\sqrt{5}-2) = 1$. □

20. Answer: 1611

Solution. Note that $n^4 + 5n^2 + 9 = n^4 - 1 + 5n^2 + 10 = (n-1)(n+1)(n^2+1) + 5(n^2+2)$.

If $n \equiv 1$ or $4 \pmod 5$, then 5 divides $n-1$ or $n+1$.

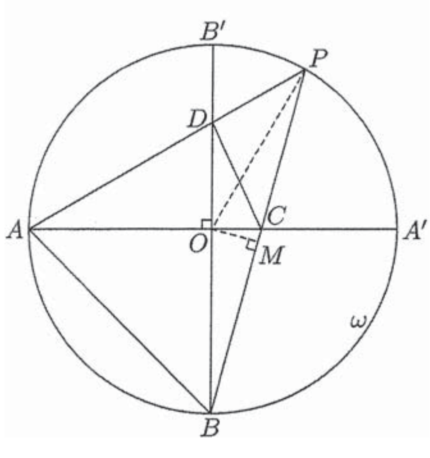
If $n \equiv 2$ or $3 \pmod 5$, then 5 divides n^2+1.

If $n \equiv 0 \pmod 5$, then 5 does not divide $(n-1)n(n^2+1)$ but divides $5(n^2+2)$, hence does not divide $n^4 + 5n^2 + 9$.

Thus, there are $2010 \div 5 = 402$ multiples of 5 from 1 to 2013. The number of integers thus required is $2013 - 402 = 1611$. □

21. Answer: 10

Solution. Since AC intersects BD at right angle, the area of the convex quadrilateral $ABCD$ is $\frac{1}{2}AC \cdot BD$. Let M be the midpoint of PB. As $\angle CAB = \angle ABD = 45^\circ$, and $\angle BCA = \angle BOM = \angle DAB$, we have $\triangle ABC$ is similar to $\triangle BDA$. Thus $AB/BD = AC/BA$. From this, we have $(ABCD) = \frac{1}{2}AC \cdot BD = \frac{1}{2}AB^2 = OA^2$ so that $OA = 10$.



□

22. Answer: 13

Solution. Given that $a_n = 2a_na_{n+1} + 3a_{n+1}$ we obtain $a_{n+1} = \dfrac{a_n}{2a_n+3}$. Thus we have $\dfrac{1}{a_{n+1}} = 2 + \dfrac{3}{a_n}$. We thus have $\dfrac{1}{a_{n+1}} + 1 = 3\left(1 + \dfrac{1}{a_n}\right)$ for all $n = 1, 2, 3, \cdots$. Letting $b_n = 1 + \dfrac{1}{a_n}$, it is clear that the sequence $\{b_n\}$ follows a geometric progression with first term $b_1 = 1 + \dfrac{1}{a_1} = 3$, and common ratio 3. Thus, for $n = 1, 2, 3, \cdots$, $b_n = 1 + \dfrac{1}{a_n} = 3^n$ for $n = 1, 2, 3, \cdots$.

Let $f(n) = \displaystyle\sum_{k=1}^{n} \frac{1}{n + \log_3 b_k} = \sum_{k=1}^{n} \frac{1}{n+k} > \frac{m}{24}$, $n = 2, 3, 4, \cdots$. It is clear that $f(n)$ is an increasing function since

$$f(n+1) - f(n) = \frac{1}{n+1} > 0.$$

Thus $f(n)$ is a strictly increasing sequence in n. Thus the minimum value of $f(n)$ occurs when $n = 2$.

$$f(2) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{m}{24},$$

forcing $m < 14$. Thus the largest value of integer m is 13. □

23. Answer: 76

Solution. Observe that $x = 1$ is always a root of the equation

$$5x^3 - 5(p+1)x^2 + (71p - 1)x + 1 = 66p.$$

Thus this equation has all roots positive integers if and only if the two roots of the equation below are positive integers:

$$5x^2 - 5px + 66p - 1 = 0.$$

Let u, v be the two roots with $u \le v$. Then

$$u + v = p, \quad uv = (66p - 1)/5,$$

implying that

$$5uv = 66(u + v) - 1.$$

By this expression, we know that u, v are not divisible by any one of $2, 3, 11$. We also have $5uv > 66(u + v)$, implying that

$$\frac{2}{u} \ge \frac{1}{u} + \frac{1}{v} > \frac{5}{66},$$

and so $u \le 26$. As

$$v = \frac{66u - 1}{5u - 66} > 0,$$

we have $5u - 66 > 0$ and so $u \ge 14$. Since u is not a multiple of any one of $2, 3, 11$, we have

$$u \in \{17, 19, 23, 25\}.$$

As $v = \frac{66u-1}{5u-66}$, only when $u = 17$, $v = 59$ is an integer.

Thus, only when $p = u + v = 17 + 69 = 76$, the equation

$$5x^3 - 5(p+1)x^2 + (71p - 1)x + 1 = 66p$$

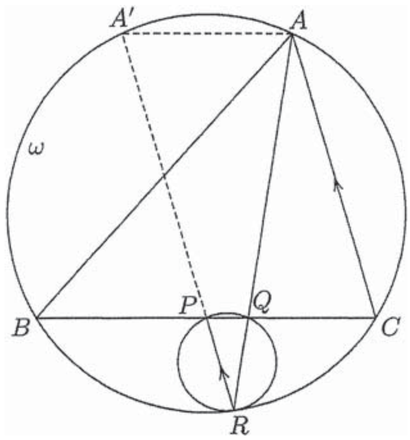
has all three roots being positive integers. □

24. Answer: 97344

Solution. First we show that M must be a square. Let $d = -a - b - c$. Then $bc - ad = bc + a(a + b + c) = (a + c)(a + b)$, $ac - bd = ac + b(a + b + c) = (b + c)(b + a)$, and $ab - cd = ab + c(a + b + c) = (c + a)(c + b)$. Therefore $M = (a + b)^2(b + c)^2(c + a)^2$. Note that $(a+b)(b+c)(c+a)$ cannot be an odd integer since two of the 3 numbers a, b, c must be of the same parity. The only squares in $(96100, 98000)$ are $311^2, 312^2, 313^2$. Since 311 and 313 are odd, the only value of M is $312^2 = 97344$. When $a = 18, b = -5, c = 6, d = -19$, it gives $M = 97344$. □

25. Answer: 64

Solution.



First by cosine rule, $\cos C = 2/7$. Reflect A about the perpendicular bisector of BC to get the point A' on ω. Then $AA'BC$ is an isosceles trapezoid with $A'A$ parallel to BC. Thus $A'A = BC - 2AC \cos C = 520 - 2 \times 455 \times 2/7 = 260$. Consider the homothety h centred at R mapping the circumcircle of PQR to ω. We have $h(Q) = A$, and $h(P) = A'$ because PQ is parallel to $A'A$. Thus A', P, R are collinear and $AA'PC$ is a parallelogram. Hence $PC = AA' = 260$, and P is the midpoint of BC. Also $PA' = CA = 455$. As $PA' \times PR = BP \times PC$, we have $455 \times PR = 260^2$ giving $PR = 1040/7$. Since the triangles PQR and $A'AR$ are similar, we have $PQ/A'A = RP/RA'$. Therefore, $PQ = 260 \times (1040/7)/(455 + 1040/7) = 64$. □