2015 World Mathematics Team Championship
Intermediate Level Individual Round

Problems	1	2	3	4	5	6	7	8
Answers	±1	2	130°	6	16	3	5	$\frac{4}{13}$
Problems	9	10	11	12	13	14		
Answers	4	$\frac{4}{3}\pi$	0	4	0	$30m^2$		
Problems	15	16	17	18		19	20	
Answers	$\frac{5}{6},\frac{7}{6},\frac{3}{2}$	$\frac{3\sqrt{3}-\pi}{\pi}$	5	23		$\frac{7}{2}\pi+\frac{\sqrt{3}}{2}$	(1, 1)	

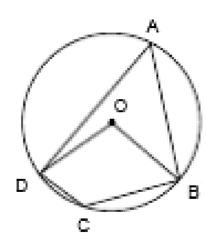
1. Suppose real number a and its reciprocal (multiplicative inverse) have the same value and real number b and its additive inverse have the same value, find all possible values for $a–b$.

Solution: ±1. Only two real numbers have the same values as their reciprocal and they are ±1.
Only one real number has the same value as its additive inverse and it is 0.
Therefore, possible values for $a–b$ are ±1.

2. Suppose real numbers x and y satisfying the equation $|x+y|+\sqrt{x-1}=0$. Find the value for $x^{2015}+y^{2016}$.

Solution: 2. From $|x+y|+\sqrt{x-1}=0$, we have $x = 1$ and $y = –1$. Therefore,
$x^{2015}+y^{2016} = 1^{2015}+(-1)^{2016} = 1+1 = 2.$

3. As shown in figure on the right, quadrilateral $ABCD$ is inscribed in circle O. If $\angle BOD = 100°$, find the degree measurement for $\angle BCD$.



Solution: 130°. Since $\angle BOD = 100°$ and the fact that, the central angle is 2 times the inscribed angle when they both extend the same arc, $\angle BAD = \angle BOD \div 2 = 50°$. Also, the two diagonally opposite angles of a circumscribed quadrilateral are supplementary. Therefore,

$\angle BCD = 180° - \angle BAD = 180° - 150° = 130°$.

4. If the product of two 2–digit numbers $\overline{x2}$ and $\overline{2y}$ is 736, find $x+y$.

Solution: 6. The units digits for these two numbers are 2 and y and the units digit of their product is 6. So, y must be 3 or 8.

$y = 3$: Then $\overline{x2} \times 23 = 736$ or $x = 3$.

$y = 8$: Then $\overline{x2} \times 28 = 736$. This is impossible since 736 is not divisible by 28.

Therefore, $x + y = 6$.

5. If the sum of all interior angles of a convex n–sided polygon is 7 times the sum of all its exterior angles, find the value of n.

Solution: 16. The sum of all exterior angles of any polygon is always 360° and the sum of all interior angles of n–sided polygon is $(n-2)\times 180°$. So $(n-2)\times 180° = 7\times 360°$ or $n = 16$.

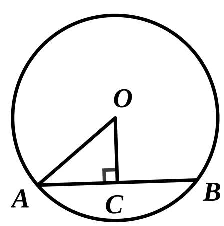
6. Find the largest integer n such that $n^{300} < 7^{200}$.

Solution: 3. Re–write $n^{300} < 7^{200}$ into $(n^3)^{100} < (7^2)^{100}$. Then $n^3 < 7^2 = 49$. The largest integer that satisfies this is $n = 3$.

7. Suppose the graph of an inverse proportion function $y = \dfrac{k}{x}$ passes through point (1, 8). Let AB be the chord of Circle O with length k and the distance from center O to AB is 3. Find the radius of Circle O.

Solution: 5. Since the function $y = \dfrac{k}{x}$ passes through (1, 8), $8 = \dfrac{k}{1}$ or $k = 8$.

As shown in the figure below, the distance from center O to AB is 3 or $OC = 3$.



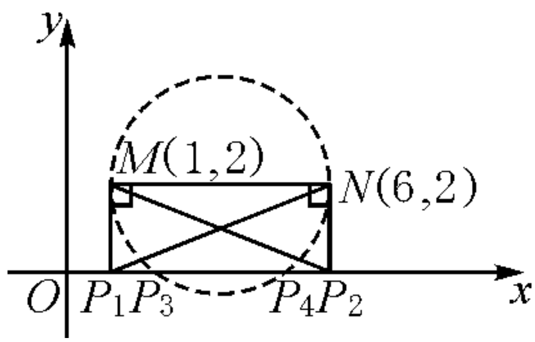
Since $AB = k = 8$, $AC = 4$ and $OA = \sqrt{3^2 + 4^2} = 5$.

8. What is the probability of selecting any three consecutive primes from a list of prime numbers that are not larger than 50 so that the sum of these three number is not a prime?

Solution: $\frac{4}{13}$**.** There are 15 prime numbers that are not greater than 50: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. If we place every 3 consecutive primes from this set and form a group, then we have a total of 13 groups: {2, 3, 5}, {3, 5, 7}, … , {37, 41, 43}, {41, 43, 47}. Now, for each group, take the sum of its three numbers. Among these 13 sums, only 4 are not prime numbers and they are 2+3+5=10, 3+5+7=15, 13+17+19=49, and 37+41+43=121. Therefore, the probability is $P = \frac{4}{13}$.

9. How many values can x take on so that the points $M(1, 2)$, $N(6, 2)$, and $P(x, 0)$ form a right triangle ΔMNP?

Solution: 4. Consider the figure below. Point P is a point on the x–axis and M and N are fixed points. In order for ΔMNP to be a right triangle, there are three possible cases.

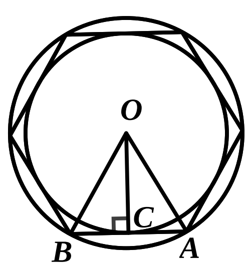


(1) $\angle M = 90$: It is easy to see that $P = (1, 0)$. In this case, $x = 1$.
(2) $\angle N = 90$: It is easy to see that $P = (6, 0)$. In this case, $x = 6$.
(3) $\angle P = 90$: P must be on the circle with MN as its diameter and the radius of this circle must be $\frac{6-1}{2} = 2.5$. Because the circle's center is 2 units away from the x–axis which is shorter than the circle's radius, this circle intersects the x–axis at two points which means there are two possible points P that satisfy the condition.

Combining these 3 cases, there is a total of 4 values for x.

10. If the radius of the inscribed circle of a regular hexagon is 1, find the area of this hexagon's circumcircle.

Solution: $\frac{4}{3}\pi$**.** Consider the figure below. Let point O be the center of the inscribed circle.

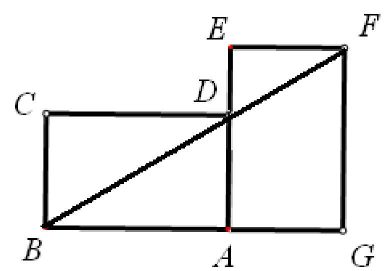


Connect OA, OB, and AB and let OC be the height of the triangle to AB. Then OC would be the radius of the inscribed circle and OB is the radius of the circumcircle. Because $\angle AOB = 360° \div 6 = 60°$ and $OA = OB$, so ΔOAB is an equilateral triangle and that means $\angle CBO = 60°$. Since $OC = 1$ and $\triangle BOC$ is a right triangle, OB = $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ and the area of the circumcircle is $\pi\left(\frac{2\sqrt{3}}{3}\right)^2 = \frac{4}{3}\pi$.

11. Find the root(s) for equation $\left(x-\frac{1}{1-x}\right)^2 \div \frac{x^2-x+1}{x^2-2x+1} = 1$.

Solution: 0. $\left(x-\frac{1}{1-x}\right)^2 \div \frac{x^2-x+1}{x^2-2x+1} = 1$ or $\left(x-\frac{1}{1-x}\right)^2 \times \frac{x^2-2x+1}{x^2-x+1} = 1$ or $\left(\frac{-x^2+x-1}{1-x}\right)^2 \times \frac{(x-1)^2}{x^2-x+1} = 1$. This can be simplified into $x^2-x+1=1$ in which $x = 0$ or 1. However, $x = 1$ leads to zero in the denominator. Therefore, $x = 0$.

12. Rotate rectangle $ABCD$ clockwise 90° around point A to $AEFG$ position as shown in the figure below. If the area of ΔBCD is $6+2\sqrt{5}$, find the area of ΔDEF.



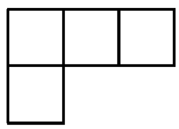
Solution: 4. Since $AD // GF$, so $\triangle BDA \backsim \triangle BFG$ which means $\frac{AD}{GF} = \frac{BA}{BG}$. ①

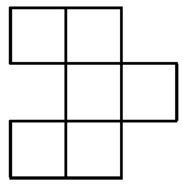
Let the lengths of the rectangles $AB=CD=AE=GF=a$ and the widths $BC=AD=AG=EF=b$. Then ① becomes $\frac{b}{a} = \frac{a}{a+b}$ and $\frac{a}{b} = \frac{\sqrt{5}+1}{2}$ (negative solution had been discarded). Therefore, the proportion between the area of $\triangle BCD$ and area of $\triangle DEF$ is: $\frac{S_{\Delta BCD}}{S_{\Delta DEF}} = \frac{\frac{1}{2}ab}{\frac{1}{2}b(a-b)} = \frac{\sqrt{5}+1}{\sqrt{5}-1} = \frac{3+\sqrt{5}}{2}$. Since the area of $\triangle BCD$ is $6+3\sqrt{5}$, so the area of $\triangle DEF$ is $\left(6+2\sqrt{5}\right) \div \frac{3+\sqrt{5}}{2} = 4$.

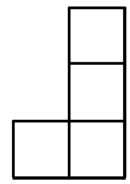
13. Suppose straight line $y = kx+b$ intersects hyperbola $y = \frac{k}{x}$ at points $A(x_1, y_1)$ and $B(x_2, y_2)$. Find the value of $x_1^3x_2 + x_1^2 + x_2^2 + x_1x_2^3$.

Solution: 0. Set $kx+b=\dfrac{k}{x}$ and simplify it to $kx^2+bx-k=0$. Since x_1 and x_2 are the roots, $x_1x_2=-1$. Therefore, $x_1^3x_2+x_1^2+x_2^2+x_1x_2^3=(x_1x_2+1)(x_1^2+x_2^2)$ $=(-1+1)(x_1^2+x_2^2)=0$.

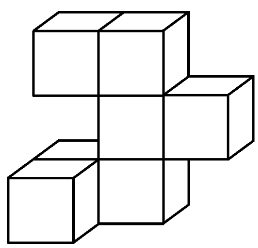
14. A geometric solid is composed by pasting many identical small cubes together side to side. The figures below show the different views of this solid. Let m be the edge length of these cubes. Find the surface area of this solid.


Top View


Front View


Right Side View

Solution: 30*m*². Based on these three figures on top, we can see that this geometric solid should look like the figure blow.



Therefore, the surface area of this solid is $(5+5+6+6+4+4)m^2=30m^2$.

15. If $[2x+1]=3x-\dfrac{1}{2}$, find x.

(Note: [x] represents the largest integer that is not greater than x.)

Solution: $\dfrac{5}{6},\dfrac{7}{6},\dfrac{3}{2}$**.** From the definition of [ ], we have

$$\begin{cases} 3x-\dfrac{1}{2} \text{ is an integer} & (1) \\ 3x-\dfrac{1}{2}\le 2x+1<3x-\dfrac{1}{2}+1, & (2) \end{cases}$$

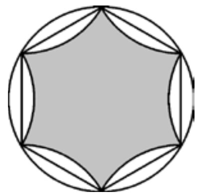
Solving the inequalities from ②, we have $\dfrac{1}{2}<x\le\dfrac{3}{2}$, (3)

Let $3x-\dfrac{1}{2}=a$ where a is an integer. Then $x=\dfrac{2a+1}{6}$, (4)

Substitute (4) into (3), we have $\dfrac{1}{2}<\dfrac{2a+1}{6}\le\dfrac{3}{2}$ or $1<a\le 4$.

Substitute a = 2, 3, 4 into (4) and we have $x = \frac{5}{6}, \frac{7}{6}, \frac{3}{2}$.

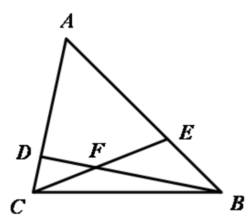
16. As shown in the figure below, the outside circle is the circumcircle of radius 2 of a regular hexagon. The shaded 6–pointed star region is formed by using each side of this hexagon as axis of symmetry and flip the corresponding arc 180°. Find the proportion of the area of the shaded region to the area of the circumcircle.



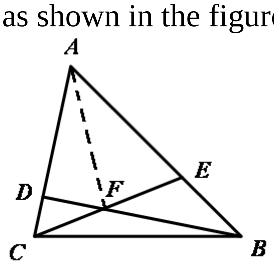
Solution: $\frac{3\sqrt{3}-\pi}{\pi}$**.** The areas of the circle and hexagon are 4π and $6\times\frac{1}{2}\times 2\times\sqrt{2^2-1^2}=6\sqrt{3}$, respectively. Hence, the area of the region between the circle and hexagon is $4\pi-6\sqrt{3}$ and the area of the shaded region is $4\pi-2(4\pi-6\sqrt{3})=12\sqrt{3}-4\pi$.

Therefore, the proportion is $\frac{12\sqrt{3}-4\pi}{4\pi}=\frac{3\sqrt{3}-\pi}{\pi}$.

17. As shown in the figure below, points D and E are on sides AC and AB of ΔABC, respectively, and straight lines DB and EC intersect at point F. If the areas of ΔCDF, ΔBEF, and $AEFD$ are 3, 4, and $\frac{204}{13}$, respectively, find the area of ΔBCF.



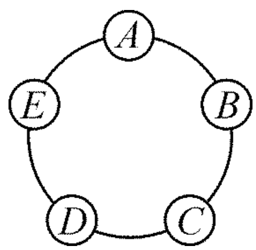
Solution: 5. Connect AF as shown in the figure below.



Then $\frac{S_{\Delta AEF}+S_{\Delta BEF}}{S_{\Delta ADF}}=\frac{S_{\Delta ABF}}{S_{\Delta ADF}}=\frac{BF}{DF}=\frac{S_{\Delta BCF}}{S_{\Delta CDF}}$ and $\frac{S_{\Delta ADF}+S_{\Delta CDF}}{S_{\Delta AEF}}=\frac{S_{\Delta ACF}}{S_{\Delta AEF}}=\frac{CF}{EF}=\frac{S_{\Delta BCF}}{S_{\Delta BEF}}$.

Let $S_{\Delta BCF}=x$. Then $\dfrac{S_{\Delta AEF}+4}{\frac{204}{13}-S_{\Delta AEF}}=\dfrac{x}{3}$ and $\dfrac{\frac{204}{13}-S_{\Delta AEF}+3}{S_{\Delta AEF}}=\dfrac{x}{4}$ since $S_{\Delta AEF}+S_{\Delta ADF}=S_{AEFD}=\dfrac{204}{13}$. Solving these two equations and we have $S_{\Delta AEF}=\dfrac{108}{13}$ and $\Delta BCF = x = 5$.

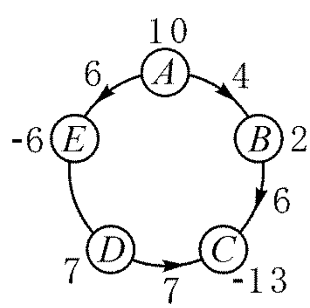
18. As shown in the figure below, five people *A, B, C, D,* and *E* are sitting around a circular table each holding 70, 62, 47, 67, and 54 marbles, respectively. How many times must they "adjust" so that each of them holds the same number of marbles? ("Adjust" once means one marble is transferred from one person to the person sitting next to him.)



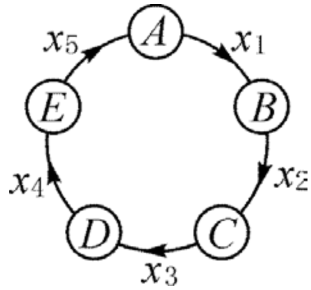
Solution: 23. In order for everyone has the same number of marbles, each one must hold (70＋62＋47＋67＋54）÷ 5＝60 marbles .

	A	*B*	*C*	*D*	*E*
Before "Adjust"	70	62	47	67	54
After "Adjust"	60	60	60	60	60
Net Required	10	2	－13	7	－6

Based on above table, we can follow the below scheme to "adjust" so that each person will have the same number of marbles and it would take 6+4+6+7 = 23 adjusts.



(Alternate Solutions): Suppose A adjusts x_1 marbles to B, B adjusts x_2 marbles to C, C adjusts x_3 marbles to D, D adjusts x_4 marbles to E, and E adjusts x_5 marbles to A where x_i would be a negative number if the arrow changes direction.



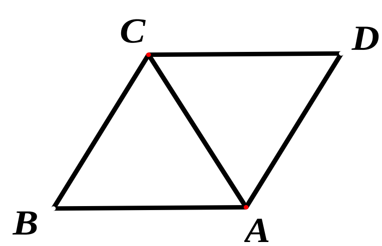
According to the problem, each one should hold 70＋62＋47＋67＋54）÷5＝60 marbles since each person holds the same number of marbles. In this case, we should have the following set of equations: $\begin{cases} 70-x_1+x_5=60, \\ 62+x_1-x_2=60, \\ 47+x_2-x_3=60, \\ 67+x_3-x_4=60, \\ 54+x_4-x_5=60. \end{cases}$ which yields solutions $\begin{cases} x_2=x_1+2, \\ x_3=x_1-11, \\ x_4=x_1-4, \\ x_5=x_1-10. \end{cases}$

The total number of "adjusts" must be made is:

$$|x_1|+|x_2|+|x_3|+|x_4|+|x_5|=|x_1|+|x_1+2|+|x_1-11|+|x_1-4|+|x_1-10| \qquad (*)$$

This takes on smallest value when $x_1=4$ in which case, 4＋6＋7＋0＋6＝23.

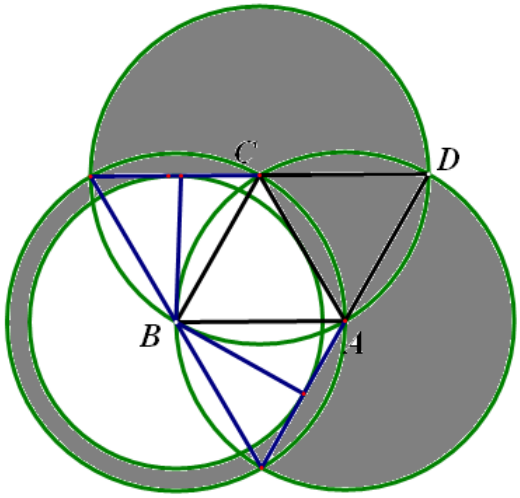
19. As shown in the figure below, ΔABC and ΔACD are both equilateral triangles of side length 1. Fix the position of ΔABC and rotate ΔACD one revolution around ΔABC using point *C* as anchor. Do the same thing using *B* and *A* as anchor. Find the total area that is swept by line segment *AC* after all three revolutions.



Solution: $\frac{7}{2}\pi+\frac{\sqrt{3}}{2}$**.** See the figure below that shows the swept area.

$$S=\left(3\pi\times 1^2+4\times\frac{1}{2}\times 1\times 1\times\sin 60^\circ\right)-\left[2\times\frac{1}{2}\times 1\times 1\times\sin 60^\circ+\frac{360-120}{360}\pi\times\left(\frac{\sqrt{3}}{2}\right)^2\right]$$

$$=3\pi\times 1^2+2\times\frac{1}{2}\times 1\times 1\times\sin 60^\circ-\frac{360-120}{360}\pi\times\left(\frac{\sqrt{3}}{2}\right)^2=\frac{7}{2}\pi+\frac{\sqrt{3}}{2}.$$



20. Suppose the parabola $y=-x^2+bx$ passes through $B(2, 0)$. Let C be a point on the parabola's axis of symmetry and $A = (4, 4)$. Find the coordinates of C so that the length $AC+BC$ is smallest.

Solution: (1, 1). Since parabola $y=-x^2+bx$ passes through point $B(2, 0)$, so $-2^2+2b = 0$ or $b=2$. Therefore, the expression for the parabola is $y=-x^2+2x$ and its axis of symmetry is $x = 1$.

Let A' be the image of $A(4, 4)$ reflect over $x = 1$. Then $A'(-2, 4)$. To pick point C on the axis of symmetry so that the length $AC+BC$ is smallest, C must be the intersection of the straight line $A'B$ which is $y = -x+2$ and $x = 1$. Therefore, the coordinates for C must be (1, 1).