2015 World Mathematics Team Championship
Junior Level Team Round
Solutions

Problems	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Answers	6047	22500	22.26	2015	120	25	21
Problems	**8**	**9**	**10**	**11**	**12**	**13**	**14**
Answers	10	150	$\frac{39}{4}$	864	1	153	890
Problems	**15**	**16**	**17**	**18**	**19**	**20**	
Answers	1479	27	4	2	75	(4, 4) or (5, 5)	

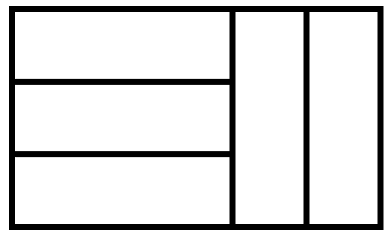
T–1. Find the sum of the digits of the quotient $\underbrace{111\cdots11}_{2015}\underbrace{222\cdots22}_{2015}\underbrace{333\cdots33}_{2015} \div \underbrace{333\cdots33}_{2015}$.

Solution: 6047.

$$\begin{aligned}
&\underbrace{111\cdots11}_{2015}\underbrace{222\cdots22}_{2015}\underbrace{333\cdots33}_{2015} \div \underbrace{333\cdots33}_{2015}\\
&= \underbrace{111\cdots11}_{2015}\underbrace{222\cdots22}_{2015}\underbrace{333\cdots33}_{2015} \div (\underbrace{111\cdots11}_{2015}\times 3)\\
&= \underbrace{111\cdots11}_{2015}\underbrace{222\cdots22}_{2015}\underbrace{333\cdots33}_{2015} \div \underbrace{111\cdots11}_{2015} \div 3\\
&= 1\underbrace{000\cdots00}_{2014}2\underbrace{000\cdots00}_{2014}3 \div 3\\
&= \underbrace{333\cdots33}_{2014}4\underbrace{000\cdots00}_{2014}1
\end{aligned}$$

Therefore, the answer is $3\times 2014+4+1=6047$.

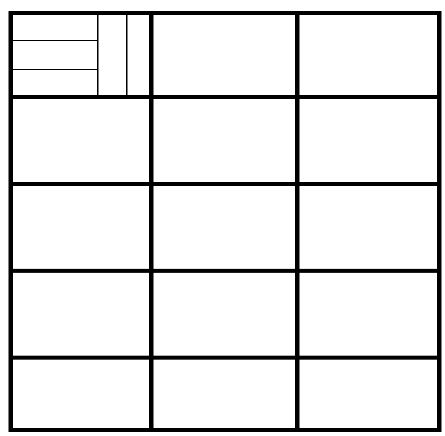
T–2. Use five small identical rectangles to form a large rectangle with perimeter of 160 as shown in the figure below. What is the area of the smallest square that can be formed using this kind of large rectangles?



Solution: 22500. Consider the figure below. Let the width of each of the small rectangles be x. Then its length is $3x$ and the perimeter of the large rectangle is

$$(x+x+x+3x+x+x)\times 2=16x.$$

Hence, $16x = 160$ or $x = 10$. Therefore, the width of the large rectangle is $3\times10 = 30$ and its length is $30+2\times10 = 50$. In order to use 30×50 rectangles to form a square, we need at least 15 of them as shown in the figure below. Therefore, the smallest square that satisfies the condition of the problem is $15\times(5\times10\times30) = 22500$.



T–3. Given two single decimal numbers $4.\square$ and $5.\Delta$ where $\square$ and Δ represent digits from 1 to 9. If we round the product $(4.\square\times 5.\Delta)$ into a single decimal number, this product becomes 22.3. Find the actual product before rounding.

Solution: 22.26. According to the problem, $22.25\le(4+\frac{a}{10})(5+\frac{b}{10})<22.35$, $\square = a$ and $\Delta = b$.

Solving these inequalities, we have $225-ab\le 50a+40b<235-ab$. However, a and b are digits from 1 to 9, so $1\le ab\le 81$. Hence, $144\le 50a+40b<234$.

Because $40(a+b)<50a+40b<50(a+b)$, so $3\le a+b\le 5$.

If $a+b=3$ or $a+b=4$, there are no solutions for a and b.

If $a+b=5$, then $a = 2$ and $b = 3$. In this case, the product $4.2\times5.3 = 22.26$ which can be rounded to 22.3. ($4.3\times5.2 = 22.36$ is too large)

T–4. For each binary number (base 2) with 2015 as the sum of its digits, rewrite this number in base 8 and compute the sum of its digits. Among all such possible base 8 numbers and their sums of digits, what is the minimum sum?

Solution: 2015. In order to minimize the sum of digits of these base 8 numbers, we have to consider binary numbers with three neighboring digits of '001' because this would make the base 8 number of '1' which is the smallest. Since the sum of digits of the binary number is 2015, all it needs is to place 2015 groups of '001' together. In this case, the minimum sum of the re–written base 8 number is $1 \times 2015 = 2015$.

T–5. Given a stack of 5 cards each is written with a different number from 0, 1, 2, 4, or 5. Choose 1 or 2 or 3 or 4 or 5 cards from this stack of cards and form a 1–digit, 2–digit, 3–digit, 4–digit, or 5–digit number. Discard any number formed in this manner that contains 2, 0, 1, and 5 as digits. If the remaining numbers are ordered from large to small, what is the 123rd number?

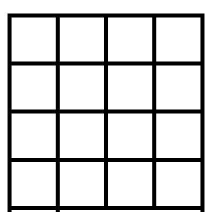
Solution: 120. Since any number formed cannot contain the four digits 2, 0, 1, and 5, there is no need to consider any 5–digit numbers.

As for 4–digit numbers, there is a total of $4 \times 4 \times 3 \times 2 = 96$ choices since 0 cannot be the leading digit. However, there is a total of $3 \times 3 \times 2 \times 1 = 18$ possible numbers that have all four digits 2, 0, 1, and 5 as digits (again, 0 cannot be the leading digit). Therefore, there can only be $96 - 18 = 78$ 4–digit numbers.

As for 3–digit numbers, there is a total of $4 \times 4 \times 3 = 48$ choices.

Since $78 + 48 - 123 = 3$, so the 123rd number is a 3–digit number and it is the 4th 3–digit number if we order the numbers from small to large which means 102, 104, 105, 120,… Therefore, the 123rd number is 120.

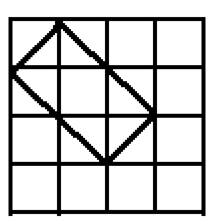
T–6. Consider the 4×4 grid square in the figure below. Any intersections of horizontal and vertical lines are called "grid points" which include the points on the perimeter. How many rectangles (including squares) of area 4 can be formed using these "grid points" as vertices?



Solution: 25. 1×4 rectangles: 8

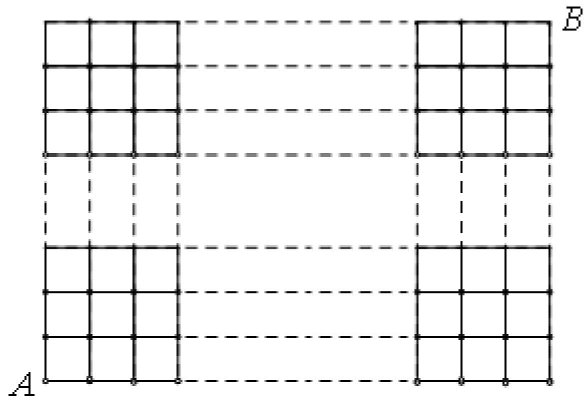
2×2 rectangles: $3 \times 3 = 9$.

Rectangles of the shape as shown in the figure below: 8.

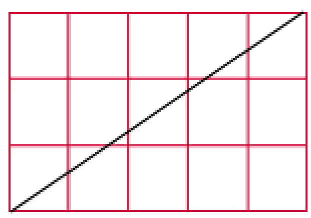


Therefore, there is a total of $8 + 9 + 8 = 25$ rectangles of area 4.

T–7. The figure below shows a 60×100 (60 rows and 100 columns) grid figure. Each intersection point of vertical and horizontal lines is called a "grid point." Let A and B be the diagonally opposite corner grid points. If A and B are connected with a straight line, how many grid points (counting A and B) would this straight passes through?



Solution: 21. Since 60:100 = 3:5, we can first consider similar rectangle of smaller scale of 3×5 as shown in the figure below. Now draw a diagonal.



This diagonal only passes through two grid points (the two corner points) which means, horizontally, every time the line passes by 5 little squares, it passes through one grid point. Similarly, vertically, every time the line passes by 3 little squares, it also passes through one grid point. Therefore, according to the problem, this diagonal line passes through $100 \div 5+1 = 60 \div 3+1 = 21$ grid points.

T–8. If the seven digit number $\overline{20xy15z}$ is divisible by 792, find $x+y+z$.

Solution: 10. Because $792=3\times3\times8\times11$, so $\overline{20xy15z}$ is divisible by 8, 9, and 11.

(1) Property when a number is divisible by 8: The last three digits must by divisible by 8. In order for $\overline{15z}$ to be divisible by 8, the number must be 152. Therefore, $z = 2$.

(2) Property when a number is divisible by 9: The sum of all digits must be divisible by 9. So, $2+x+y+1+5+2 = x+y+10$ must be divisible by 9. Therefore, $x+y = 8$ or $x+y = 17$.

(3) Property when a number is divisible by 11: The difference between the sum of the odd positioned digits and the sum of the even positioned digits is a multiple of 11. That means, $2+x+1+z = y+5$ or $2+x+1+z = y+5\pm11$. Combine these with $x+y = 8$ or $x+y = 17$, we have $x = y$ or $x = y\pm11$ which means $x = y = 4$.

Therefore, $x+y+z = 2+4+4 = 10$.

T–9. Suppose the sum of the dividend, divisor, quotient, and remainder of one particular division is 181 and its divisor is the same as the product of its quotient and remainder. Find the dividend.

Solution: 150. Let E = Dividend, D = Divisor, Q = Quotient, and R = Remainder. According to the problem, E = QD+R and D = Q×R. So, (Q+D+E+R) = (Q+QR+(QD+R)+R) = 181 or (181–2R–QR–Q) ÷ Q = D = QR.

R = 1: (181–2–Q–Q)÷Q = Q or $Q^2 + 2Q - 179 = 0$ … No positive integer solution.

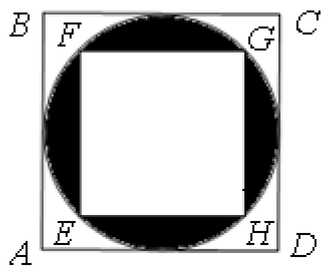
R = 2: (181–4–2Q–Q)÷Q = 2Q or $2Q^2 + 3Q - 177 = 0$ … No positive integer solution.

R = 3: (181–6–3Q–Q)÷Q = 3Q or $3Q^2 + 4Q - 175 = 0 = (Q-7)(3Q+25)$. So, Q = 7, D = 21, and E = 150.

R > 3: Use similar method to show that there is no positive integer solution for Q.

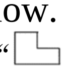
Therefore, the dividend is 150.

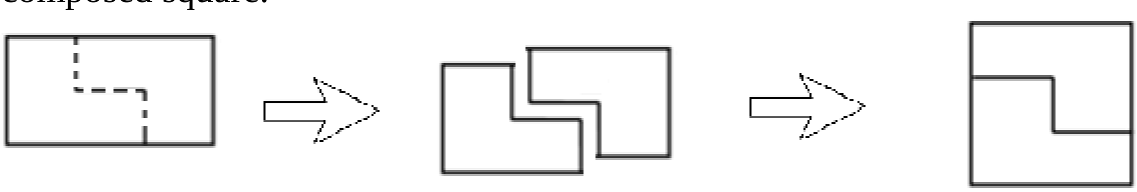
T–10. As shown in the figure below, the circle is the circle with the largest area inside square *ABCD* and rectangle *EFGH* is the square with the largest area inside that circle. If the difference of the areas of these two squares is 16 and the difference of their perimeters is 8, find the area of the shaded region. (use π = 3)



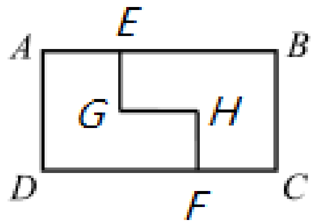
Solution: $\frac{39}{4}$. Let A and a be the side lengths of the large square and the small square, respectively. Then $\begin{cases} 4A-4a=8, \\ A^2-a^2=16, \end{cases}$ or $\begin{cases} A-a=2, \\ A+a=8. \end{cases}$ Solve this and we have $A = 5$ and $a = 3$.

According to the problem, $S_{shaded} = S_{circle} - S_{square} = \pi(\frac{A}{2})^2 - a^2 = 3\times(\frac{5}{2})^2 - 3^2 = \frac{39}{4}$.

T–11. Consider the figures below. Suppose a rectangle is cutting up into two identical pieces in the shape of "" and that these two pieces of "" can be composed together to form a square. If the sum of the perimeters of these two pieces is 2016 more than the perimeter of the original rectangle, find the side length of the composed square.



Solution: 864. As shown in the figure below, because rectangle can be recomposed into a square after cutting, $AE = GH = FC$ and $EG = HF$.



Also, $2AE$ = the side length of the square = $3EG$.

Let the side length of the square be a. Then $AE = \frac{1}{2}a$ and $AB = AE + EB = (\frac{1}{2}+1)a = \frac{3}{2}a$.

So, $EG = AE\times 2 \div 3 = \frac{1}{3}a$ and $AD = EG\times 2 = \frac{2}{3}a$. Since the perimeter of this new figure is 2016 longer than the perimeter of the original rectangle, $2(EG+GH+HF) = 2016$ or $EG+GH+HF = EG+AE+EG = 1008$. This means $\frac{1}{3}a + \frac{1}{2}a + \frac{1}{3}a = 1008$ or $a = 864$.

T–12. Given that both $\overline{MA}$ and $\overline{TH}$ are 2–digit numbers where different letters represent different digits. If $\overline{MA} + \overline{TH} = 89$, find the units digit of $(M+A+T+H)^{2016}$. (Note: n^{2016} means the product of 2016 n's)

Solution: 1. When both A and H take on the maximum value of 9, $A+H = 18$. Since the units digit of $\overline{MA}+\overline{TH}$ is 9，there is no carrying in $A+H$. So, $A+H = 9$ and $M+T = 8$. Hence, $(M+A+T+H)^{2016} = 17^{2016}$. Since the units digit of 7^1 is 1, 7^2 is 9, 7^3 is 3, 7^4 is 1, and 7^5 is 7, this pattern repeats every 4 numbers. However $2016\div 4 = 504$ and the units digits of 17^{2016} and 7^{2016} and 7^4 are the same. Therefore, the units digit of $(M+A+T+H)^{2016}$ is 1.

T–13. Observe the pattern of the following table. If the last row has only one number which we call it x, find the number of factors of x.

1,	**2,**	**3,**	**4,**	**5,**	**…,**	**95,**	**96,**	**97,**	**98,**	**99**
	4,	**6,**	**8,**	**10,**	**…,**	**190,**	**192,**	**194,**	**196**	
		12,	**16,**	**20,**	**…,**	**380,**	**384,**	**388**		
		…		**…**	**…**	**…**		**…**		
				…	**…**	**…**				
					x					

Solution: 153. It is easy to see from the table that the first row has 99 numbers, the second row has 97 numbers, and so on until the 50th row has only 1 number so there is a total of 50 rows. Consider the first row. If we reverse the first row and add each number by the column to the original first row, we would get 99 100's with the sum of the two middle numbers being $100 = 100\times 2^0$. Do the same to the 2nd row and we have 97 200's with the sum of two middle numbers being 200 = 100×2^1. Follow this pattern, the last row, being the 50th row, should have the sum of the two middle numbers 100×2^{49}. Hence, $x = 100\times2^{48} = 2^{50}\times5^2$. This number should have (50+1) × (2+1) = 51×3 = 153 factors.

T–14. A store is selling pears and apples. The number of apples in this store is 10 less than 3 times the number of pears it has during the first day. If this store can sell 30 pears and 70 apples every day, 8 days later, the number of apples in the store is 30 more than 5 times the number of pears it has. How many apples does it have originally?

Solution: 890. Suppose, originally, a store has a apples and b pears for sales. Then $a = 3b–10$. Eight days later, the store has $(a–8\times70) = (a–560) = 5(b–240)+30$ apples or $a = 5b–610$. So, $3b–10 = 5b–610$ or $b = 300$. Therefore, $a = 3\times300–10 = 890$ apples

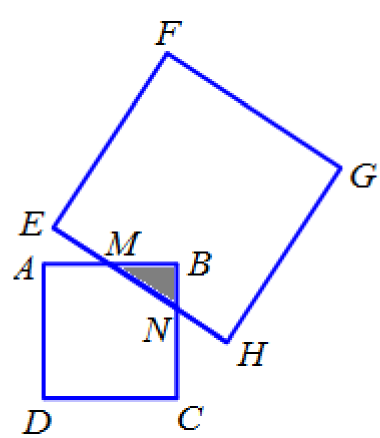
T–15. How many different values does $[\frac{n}{2}]+[\frac{n}{3}]+[\frac{n}{5}]$ take if n is a natural number that takes on values from 1 to 2015? (Note: $[x]$ represents the largest integer not greater than x).

Solution: 1479. Because 30 is the Lowest Common Multiple of 2, 3, and 5, let $n = 30k+p$ where p is the remainder when n is divided by 30. Then

$$[\frac{n}{2}]+[\frac{n}{3}]+[\frac{n}{5}]=(15k+10k+6k)+[\frac{p}{2}]+[\frac{p}{3}]+[\frac{p}{5}]=31k+[\frac{p}{2}]+[\frac{p}{3}]+[\frac{p}{5}].$$

When $p=0,1,2,\cdots,29$, $[\frac{p}{2}]+[\frac{p}{3}]+[\frac{p}{5}]$ has a total of 22 values. That means, this expression has 20 values for every 30 p's starting with $p = 0$. Since there are 2016 integers from 0 to 2015 and $2016\div30=67\cdots\cdots6$, so $[\frac{n}{2}]+[\frac{n}{3}]+[\frac{n}{5}]$ has a total of $67\times22+5=1479$ different values.

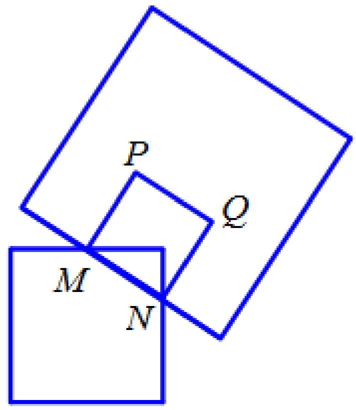
T–16. As shown in the figure below, square $ABCD$ and square $EFGH$ overlapped in the shaded region. Suppose $AM = MB$, $CN = 2NB$, and $EM = MN = NH$. If this overlapped region has an area of 1, find the difference in areas of the two non–overlapped regions of these squares.



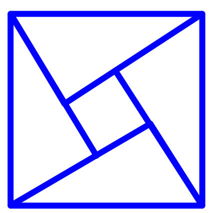
Solution: 27. The difference in areas between two non–overlapped regions of two squares is equal to the difference in areas of these two squares. Hence, to answer this problem, we only need to find the areas of these two squares.

Area of Square *ABCD*: Because *M* is the midpoint of *AB* and *N* is the one–third points of *BC*, so the area of *ABCD* is $2\times2\times3 = 12$ times the area of $\triangle BNM$ which is 1. Therefore, the area of Square *ABCD* is 12.

Area of Square *EFGH*: Consider the figure below. Because Points *M* and *N* trisect *EH*, so the area of Square *EFGH* is 9 times the area of small Square *MPQN*.



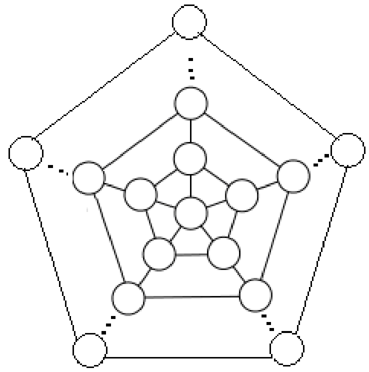
We can further divide Square *MPQN* as shown in the figure below. Suppose the length of the long side of the triangle is $3x$. Then the length of the short side is $2x$ and the area of the square at the center is x^2.



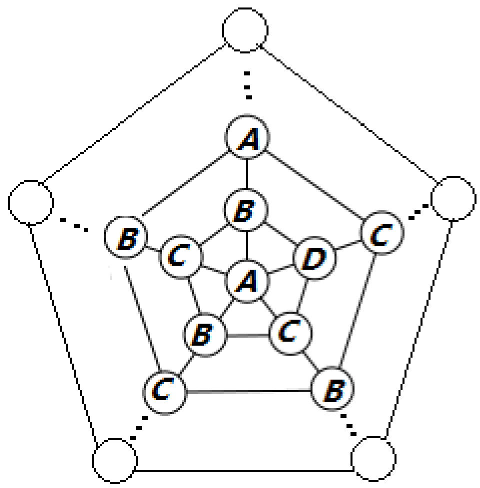
Since the area of the triangle is 1, so $(3x)(2x)\div2 = 1$ or $x^2 = 1/3$. Hence, the area of Square *MPQN* is $4+1/3 = 13/3$ and the area of Square EFGH is $9\times13\div3=39$.

Therefore, the difference between the areas of these two squares is $39–12 = 27$.

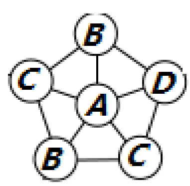
T–17. Each circle in the figure below is painted with onr color. How many different colors are needed to paint these circles so that any two circles that are connected by a straight line will have different colors?



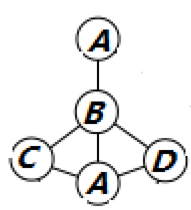
Solution: 4. As shown in the figure below, it takes at least 4 colors to color the circles starting from the center and spread out in filling in the circles (each letter represents a different color).



Case (1): Consider the center circle that is connected to 5 circles that surround it. As show in the figure below, it takes at least 4 colors to color them.

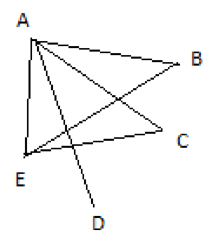


Case (2): Consider the off–center circle that is connected to 4 circles that surround it. As show in the figure below, it also takes at least 4 colors to color them.



T–18. Five people A, B, C, D, and E are involved in a gift exchange program under the following two conditions: (1) If a person receives a gift, he must return a gift to the original giver, (2) Gift exchange between two people can only happen 0 or 1 time. If A receives 4 gifts, B receives 2 gifts, D receives 1 gift, and E receives 3 gifts, how many gifts C would receive based on the above two conditions.

Solution: 2. Construct a graph as below so that two people (points) are connected by a line if they have exchanged gifts. Because A had received 4 gifts, so there are lines connecting A to each of the other 4 points. Because D had received only 1 gift, so D only connects to A. Because E received 3 gifts, there should be 3 lines connected to E. However, E cannot be connected to D, so E is connected to A, C, and E. Therefore, we have the graph below that shows C had received 2 gifts.



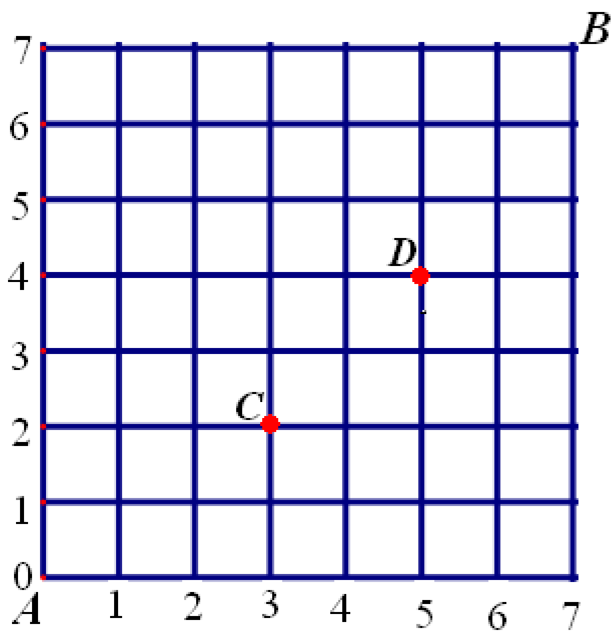
T–19. Suppose A, B, and C start from the same position running on a track at the same time trying to catch up to D who is already running at the time. Given that the running speeds of A, B, and C are 100 meters/minute, 120 meters/minute, and 90 meters/minute, respectively. If it takes A 50 minutes and B 30 minutes to catch up to D, how long does it take for C to catch up to D?

Solution: 75. It takes A 50 minutes to catch up to D means when A is caught up with D, D is $100 \times 50 = 5000$ meters from the starting point.

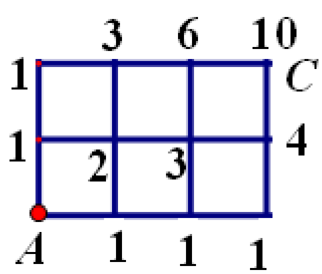
It takes B 50 minutes to catch up to D means when B is caught up with D, D is $120 \times 30 = 3600$ meters from the starting point.

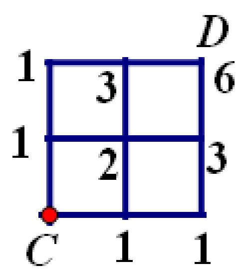
So, the speed of D is $(5000 - 3600) \div (50 - 30) = 70$ meters/minute. Therefore, D is $100 \times 50 - 70 \times 50 = 1500$ meters from the starting point when A, B, and C started to run and it would take C $1500 \div (90 - 70) = 75$ minutes to catch up to D.

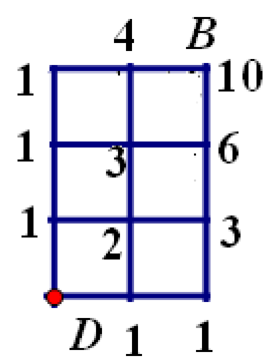
T–20. The figure below is a 7×7 grid figure and each grid point can be identified by a pair of numbers which we called them "coordinates." For examples, A(0, 0), B(7, 7), C(3, 2), and D(5, 4). Suppose an ant starts from A crawling either up or to the right passing through points C and D going toward point B. If there is a "breakpoint" (a grid point where the ant cannot pass) E somewhere in this grid figure and if there is a total of 300 different paths for the ant to go from A to B, find the coordinates of point E.



Solution: (4, 4) or (5, 3). As shown in the figures below, the number of different paths (without the "breakpoints") are: 10 from A to C, 6 from C to D, and 10 from D to B.







If the "breakpoint" is between A and C, then there are 300÷10÷10=5 paths.

If the "breakpoint" is between C and D, then there are 300÷10÷10=3 paths.

If the "breakpoint" is between D and B, then there are 300÷10÷6=5 paths.

Consider these three cases, in order to satisfy the condition, E must be between C and D. In fact, there can only be two such E as shown in the two figures below: E(4, 4) and E(5,3).