2015 World Mathematics Team Championship
Intermediate Level Team Round
Solutions

Problems	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Answers	44	180	2	$\frac{25}{6}$	14	$3\sqrt[3]{12}$	$806\sqrt{5}$
Problems	**8**	**9**	**10**	**11**	**12**	**13**	**14**
Answers	15	25	$\frac{2015}{1008}$	2	2	13	$14+2\sqrt{3}+\sqrt{5}$
Problems	**15**	**16**	**17**	**18**	**19**	**20**	
Answers	16	1344	3	$\frac{3+\sqrt{6}+\sqrt{15}}{2}$	4	$\frac{\sqrt{11}-\sqrt{3}}{2}$	

T–1. Given a rectangle with an area of 910 and both its length and width are integers larger than 20. Find an integer that is closest to the length of its diagonal.

Solution: 44. Let a be the width and b be the length of this rectangle. Since both length and width are larger than 20 and $910 = 7\times13\times2\times5$, $a = 26$ and $b = 35$. Hence, the length of its diagonal is $l=\sqrt{a^2+b^2}=\sqrt{26^2+35^2}=\sqrt{1901}$. However, $43^2 = 1849$ and $44^2 = 1936$, $43 < l < 44$. Also, 43.52 = 1892.25 and 1901 > 1892.25. Therefore, the integer that is closest to the diagonal's length is 44.

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T–2. Suppose a rectangle has an area of 2016 and both its length and width are integers. Find the perimeter of such rectangle with smallest difference between its length and width.

Solution: 180. Let l = length and w = width. Then $l\times w = 2016 = 2^5\times3^2\times7$. Since both l and w are integers and we are trying to make these two numbers as close as possible, we have $l = 2^4\times3 = 48$ and $w = 2\times3\times7 = 42$. Therefore, the perimeter is $2\times(48+42) = 180$.

T–3. How many possible pairs of non–zero real numbers a and b so that there are exactly two different values among the four numbers $a+b$, $a-b$, $a\times b$, and $a\div b$?

Solution: 2. We can consider the following two cases relating to exactly two different values:

(1) These four operations give two sets of identical answers. Consider the followings:

$$\begin{cases} a+b=a-b, ① \\ a\times b=a\div b, ② \end{cases} \quad \begin{cases} a+b=a\times b, ③ \\ a-b=a\div b, ④ \end{cases} \quad \begin{cases} a+b=a\div b, ⑤ \\ a-b=a\times b, ⑥ \end{cases}$$

① implies $2b = 0$ or $b = 0$. This contradicts the fact that b is non–zero. So $\begin{cases} a+b=a-b, ① \\ a\times b=a\div b, ② \end{cases}$ does not have non–zero real solutions.

Take ③×④ (or ⑤×⑥). Then we have $a^2-b^2=a^2$. Hence $b = 0$. Contradiction.

Therefore, neither $\begin{cases} a+b=a\times b, ③ \\ a-b=a\div b, ④ \end{cases}$ nor $\begin{cases} a+b=a\div b, ⑤ \\ a-b=a\times b, ⑥ \end{cases}$ has non–zero real solutions.

(2) Exactly three of these four operations are the same.

We already know from above that $a+b$ cannot be the same as $a-b$. So, consider

$a+b=a\times b=a\div b$, ⑦ or $a-b=a\times b=a\div b$, ⑧

From ⑦, we have $\begin{cases} a=\frac{1}{2}, \\ b=-1. \end{cases}$ From ⑧, we have $\begin{cases} a=-\frac{1}{2}, \\ b=-1. \end{cases}$

Therefore, there are exactly two pairs of (a, b) satisfying the problem.

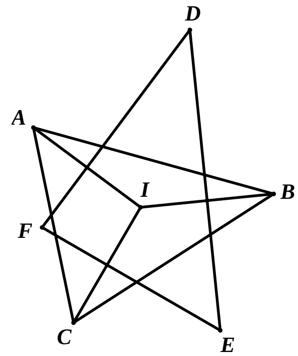
T–4. Given $a>0$ and $b>0$. If $a+b = 3$, find the smallest value for $\frac{a^2+4}{a}+\frac{b^2}{b+3}$.

Solution: $\frac{25}{6}$. From $a+b=3$, we have

$$\begin{aligned}\frac{a^2+4}{a}+\frac{b^2}{b+3} &= a+\frac{4}{a}+\frac{b^2-9+9}{b+3}\\ &= a+\frac{4}{a}+b-3+\frac{9}{b+3}\\ &= \frac{4}{a}+\frac{9}{b+3}\\ &= \left(\frac{4}{a}+\frac{9}{b+3}\right)\cdot\frac{a+(b+3)}{6}\\ &= \frac{13}{6}+\frac{2}{3}\cdot\frac{b+3}{a}+\frac{3}{2}\cdot\frac{a}{b+3}\\ &\geq \frac{13}{6}+2,\\ &= \frac{25}{6}.\end{aligned}$$

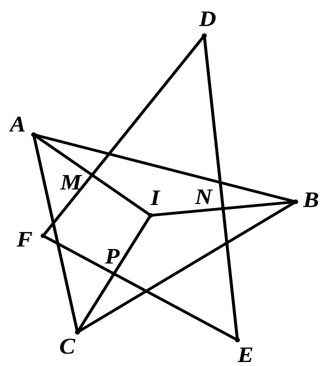
Since there is a real solution for a in the equation $\frac{a^2+4}{a}+\frac{b^2}{b+3}=\frac{a^2+4}{a}+\frac{(3-a)^2}{-a+6}=\frac{25}{6}$, the smallest values for $\frac{a^2+4}{a}+\frac{b^2}{b+3}$ is $\frac{25}{6}$.

T–5. Let I be the center of the circumcircle of ΔABC and ΔDEF is formed by using the perpendicular bisectors of IA, IB, and IC as its three sides as shown in the figure below. If $IA = 6$ and $S_{\Delta DEF} = 21$, find the perimeter of ΔDEF.



Solution: 14. As shown in the figure below, label the midpoints of IA, IB, and IC as M, N, and P, respectively. Because I is the circumcenter of $\triangle ABC$, $IA = IB = IC$. Also, $IM = IN = IP = \frac{1}{2}IA = 3$ and $IM \perp FD$, $IN \perp DE$, and $IP \perp EF$. So, point I is also the center of the inscribed circle of $\triangle DEF$ and $S_{\Delta DEF} = \frac{1}{2} \cdot IM \cdot (FD + DE + EF) = 21$.

Therefore, the perimeter of $\triangle DEF$ is $FD + DE + EF = \frac{21 \times 2}{IM} = 14$.



T–6. If $x = \sqrt[3]{16} + \sqrt[3]{12} + \sqrt[3]{9}$, find $x - \frac{1}{x^2}$.

Solution: $3\sqrt[3]{12}$ **.** Because $x = \sqrt[3]{16} + \sqrt[3]{12} + \sqrt[3]{9} = \left(\sqrt[3]{4}\right)^2 + \sqrt[3]{4 \times 3} + \left(\sqrt[3]{3}\right)^2$ ①

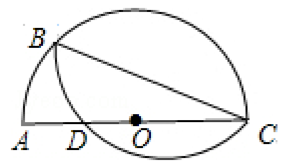
reminded the formula $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, so multiply both sides of ① by $\left(\sqrt[3]{4} - \sqrt[3]{3}\right)$. This will resulted in $x\left(\sqrt[3]{4} - \sqrt[3]{3}\right) = \left(\sqrt[3]{4} - \sqrt[3]{3}\right) \cdot \left(\sqrt[3]{4}\right)^2 + \sqrt[3]{4 \times 3} + \left(\sqrt[3]{3}\right)^2$ or

$x\left(\sqrt[3]{4} - \sqrt[3]{3}\right) = \left(\sqrt[3]{4}\right)^3 - \left(\sqrt[3]{3}\right)^3 = 1$.

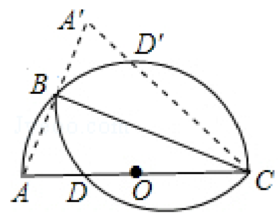
Hence, $\frac{1}{x} = \sqrt[3]{4} - \sqrt[3]{3}$ or $\frac{1}{x^2} = \left(\sqrt[3]{4} - \sqrt[3]{3}\right)^2 = \sqrt[3]{16} - 2\sqrt[3]{12} + \sqrt[3]{9}$.

Therefore, $x - \frac{1}{x^2} = \sqrt[3]{16} + \sqrt[3]{12} + \sqrt[3]{9} - \left(\sqrt[3]{16} - 2\sqrt[3]{12} + \sqrt[3]{9}\right) = 3\sqrt[3]{12}$.

T–7. Given a semi–circle *O*. Let *BC* (not its diameter) be a chord and $\widehat{BC}$ be its minor arc. As shown in the figure below, use the chord *BC* as axis of symmetry and fold the minor arc $\widehat{BC}$ over until it intersects the diameter *AC* at point *D*. If $\frac{AD}{AC}=\frac{2}{5}$ and *AC* = 2015, find the length of chord *BC*.



Solution: $806\sqrt{5}$**.** If *AC* = 2015, then *AD* = 806. Let *A′C* be the symmetric image of *AC* around the axis *BC* and *D′* is the intersection point of *A′C* and the semi–circle as shown in the figure below. Connect *AB* and *BA′*. Then *A*, *B*, and *A′* are collinear and *AB* =*A′B*, *AD* =*A′D′*= 806 and *A′C* =*AC* =2015.



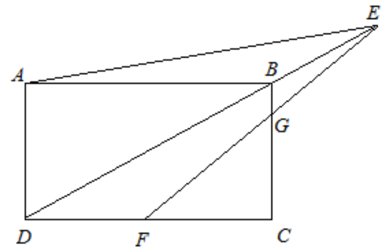
Based on Power of a Point or the Secant Theorem, $(A'B)(A'A) = (A'D')(A'C)$ or $(A'B)(2A'B) = 806\times2015 = 403^2\times10$ which means $A'B^2=403^2\times5$.

Therefore, $BC^2 = A'C^2 - A'B^2 = 2015^2-403^2\times5 = 403^2\times(5^2-5) = 403^2\times20 = 806^2\times5$ and $BC=806\sqrt{5}$.

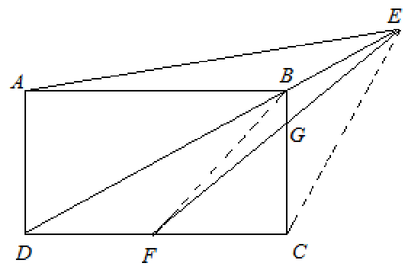
T–8. Randomly select a positive factor from 6^{2015}. If the probability of this factor happens to be a multiple of 6^{1512} is $\frac{n}{m}$ where *m* and *n* are relatively primes, find the value of $m-n$.

Solution: 15. Since $6^{2015}=2^{2015}\times3^{2015}$, all positive factors of 6^{2015} are in the form of $\begin{cases}2^a\cdot3^b,\\0\le a,b\le2015.\end{cases}$ Hence, there is a total of 2016×2016 positive factors of 6^{2015}. According to the problem, any number that is a factor of 6^{2015} and a multiple of 6^{1512} should be in the format of $\begin{cases}6^{1512}\cdot2^x\cdot3^y,\\0\le x,y\le2015-1512=503.\end{cases}$ And there are 504×504 of them. Since the probability is $\frac{n}{m}=\frac{504\times504}{2016\times2016}=\frac{1}{16}$, $m-n = 16-1 = 15$.

T–9. As shown in the figure below, point E is on the extension of rectangle $ABCD$'s diagonal DB with $DB = 2BE$. Let F be the midpoint of DC and EF intersects BC at G. If the area of ΔAEB is 100, find the area of ΔBEG.



Solution: 25. Connect CE and BF as shown in the figure below.



Because $DB = 2BE$, so $S_{\Delta BEC} = \frac{1}{2}S_{\Delta BCD} = \frac{1}{2}S_{\Delta ABD} = S_{\Delta AEB} = 100$ or $S_{\Delta ABD} = 2S_{\Delta AEB} = 200$.

Hence, the area of rectangle $ABCD$ is 400. Also, F is the midpoint of DC, so

$$S_{\Delta DEF} = S_{\Delta ECF} = \frac{S_{\Delta BCE} + S_{\Delta BCD}}{2} = \frac{100+200}{2} = 150 \text{ or } S_{\Delta BEF} = \frac{1}{1+2}S_{\Delta DEF} = 50.$$

So, $\frac{S_{\Delta BEG}}{S_{\Delta BEC}} = \frac{BG}{BC} = \frac{S_{\Delta BEF}}{S_{\Delta BEF} + S_{\Delta ECF}}$ or $\frac{S_{\Delta BEG}}{100} = \frac{50}{50+150}$.

Therefore, $S_{\Delta BEG} = 25$.

T–10. Let S_m be the area of the triangular region that is enclosed by straight lines l_1: $y = mx+2(m-1)$, l_2: $y = (m+1)x+2m$, and the x–axis where $m = 1, 2, 3, \ldots$. Find the value of $S_1 + S_2 + S_3 + \ldots + S_{2015}$.

Solution: $\frac{2015}{1008}$**.** Combine the two linear equations, we have $\begin{cases} y = mx + 2(m-1), \\ y = (m+1)x + 2m \end{cases}$ or

$\begin{cases} x = -2, \\ y = -2. \end{cases}$ That means the lines l_1 and l_2 intersect at $(-2, -2)$. Also, because l_1 intersects the x–axis at $\left(\frac{2(1-m)}{m}, 0\right)$, $S_m = \frac{1}{2}\cdot|-2|\cdot\left|\frac{2(1-m)}{m} + \frac{2m}{m+1}\right|$

$$= 2\cdot\left|\frac{1-m}{m} + \frac{m}{m+1}\right|$$

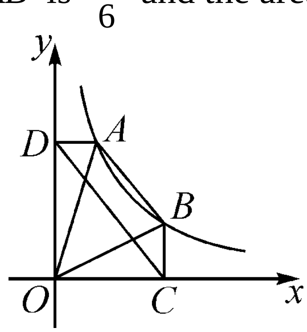
$$= 2\cdot\left|\frac{(1-m)(1+m)+m^2}{m(m+1)}\right|$$

$$= \frac{2}{m(m+1)}$$

$$= 2\left(\frac{1}{m}-\frac{1}{m+1}\right).$$

Therefore, $S_1+S_2+S_3+\cdots+S_{2015} = 2(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2015}-\frac{1}{2016}) = \frac{2015}{1008}$.

T–11. An inverse proportion function $y=\frac{k}{x}$ has two points *A* and *B* on the First Quadrant. Draw a line segment *AD* that is perpendicular to the *y*–axis at *D* and another segment *BC* that is perpendicular to *x*–axis at *C* as shown in the figure below. If the area of ΔOAB is $\frac{5}{6}$ and the area of ΔOCD is $\frac{3}{2}$, find *k*.



Solution: 2. As shown in the figure below, extend *DA* and *CB* so they intersect at point *E*.

Let *C*= (*a*, 0) and *D* = (0, *b*). Then $A = (\frac{k}{b}, b)$ and $B = (a, \frac{k}{a})$. According to the problem,

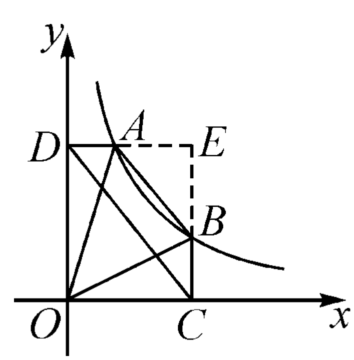
$S_{\Delta OCD} = \frac{1}{2}OC\cdot OD = \frac{1}{2}ab = \frac{3}{2}$. So, $ab = 3$. From the figure, it can be seen that

$S_{\Delta OAB} = S_{ODEC} - S_{\Delta AEB} - S_{\Delta AOD} - S_{\Delta COB}$.

$S_{ODEC} = ab$.

$S_{\Delta AOD} = S_{\Delta COB} = \frac{k}{2}$.

$S_{\Delta AEB} = \frac{1}{2}AE\cdot BE = \frac{1}{2}(a-\frac{k}{b})\cdot(b-\frac{k}{a})$.



So, $S_{\Delta OAB} = ab-\frac{1}{2}\left(a-\frac{k}{b}\right)\left(b-\frac{k}{a}\right)-\frac{k}{2}-\frac{k}{2} = \frac{ab}{2}-\frac{k^2}{2ab} = \frac{5}{6}$.

Since $ab = 3$, $\frac{3}{2}-\frac{k^2}{6}=\frac{5}{6}$ or $k = \pm 2$. However, *A* is in the First Quadrant.

Therefore, $k = 2$.

T–12. Suppose the sum of k consecutive positive integers is 2015. Find the smallest number among these k numbers.

Solution: 2. Label these k consecutive positive integers $a_1,\ a_1+1,\ a_1+2,\ \cdots,\ a_1+(k-1)$.

There sum is $\dfrac{a_1+a_1+(k-1)}{2}\cdot k = 2015$ or $(2a_1+k-1)\cdot k = 4030$ ①

So k must be a factor of $4030 = 1\times 2\times 5\times 13\times 31$ and k can only take on values of 1, 2, 5, 10, 13, 26, 31, and 62. We can change ① to $2a_1 - 1 = \dfrac{4030}{k} - k$ ②

So, a_1 decreases as k increases. Therefore, a_1 takes on minimum values when $k = 62$ which means that $a_1 \geq \dfrac{1}{2}(\dfrac{4030}{62} - 62 + 1) = 2$.

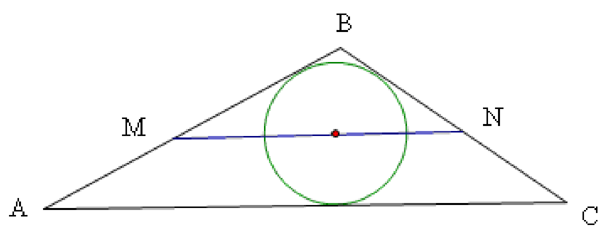
T–13. Suppose M is a positive integer and both $8M+40$ and $8M–40$ are perfect squares. Find the value of M.

Solution: 13. Suppose $8M+40 = m^2$ and $8M–40 = n^2$ where m and n are positive integers and $m > n$. Subtract these two equations and we have

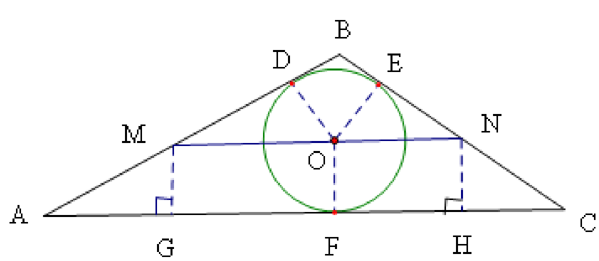
$m^2 - n^2 = (m+n)(m-n) = 80 = 2^4\times 5$.

Therefore, m^2 and n^2 are both multiples of 8 and m and n are both multiples of 4. Also, $m+n > m–n > 0$, so both $m+n$ and $m–n$ are multiple of 4. Therefore, $m+n = 20$ and $m–n = 4$ or $m = 12$ and $n = 8$. So, $M = 13$.

T–14. Consider the figure below. Suppose the inscribed circle of ΔABC has a radius of 2. Let M and N be points on AB and BC so that they are the intersections of the line that passes through the center and parallel to AC. If $MN = 7$ and $AM = 4$, find the area of the trapezoid $AMNC$.



Solution: $14+2\sqrt{3}+\sqrt{5}$ **.** Let O be the center of inscribed circle of ΔABC and points D, E, and F are the points of tangency as shown in the figure below. Connect OD, OE, and OF. From points M and N draw $MG \perp AC$ at G and $NH \perp AC$ at H.



Since $MN // AC$, $\angle A = \angle DMN$ and $MG = OF = OD$. Hence, $MG = 2$. Also, $\angle MGA = \angle MDO = 90^\circ$, so $\Delta AMG \cong \Delta MOD$ or $AM = MO = 4$. Similarly, $NC = ON = MN - MO = 3$.

Since ΔAMG is a right triangle, $AG = \sqrt{AM^2 - MG^2} = \sqrt{4^2 - 2^2} = 2\sqrt{3}$.

ΔCHN is also a right triangle, so $HC = \sqrt{NC^2 - NH^2} = \sqrt{3^2 - 2^2} = \sqrt{5}$.

That means, $AC = AG + GH + HC = 2\sqrt{3} + 7 + \sqrt{5}$.

Therefore, $S_{AMNC} = \frac{1}{2} \cdot (MN + AC) \cdot OF = \frac{1}{2} \times (7+7+2\sqrt{3}+\sqrt{5}) \times 2 = 14 + 2\sqrt{3} + \sqrt{5}$.

T–15. Suppose the equation $ax^2 + bx + c = 0$ has real solutions and its coefficients a, b, and c satisfy the following conditions:

(1) a, b, and c are positive integers;

(2) The 6–digit number $\overline{a2015b}$ is divisible by 12;

(3) c^3+3 is divisible by $c+3$.

Find the maximum value for $a+b+c$.

Solution: 16. In order for equation $y = ax^2 + bx + c$ to have real solution,

$b^2 - 4ac \geq 0$ or $ac \leq \frac{b^2}{4}$. ①

Let M = $\overline{a2015b}$. Since M is divisible by 12, M is also divisible by 3 and 4.

(1) M is divisible by 4: Then the last two digits of M $\overline{5b}$ is divisible by 4. So, $b = 2$ or 6. When $b = 2$, ① implies $ac \leq \frac{b^2}{4} = 1$, so $a = c = 1$ since a and c are positive integers. However, in this case, the 6–digit number

$\overline{a2015b} = 120152$ which cannot be divisible by 3. So, b must be 6 which means $ac \leq 9$ ②

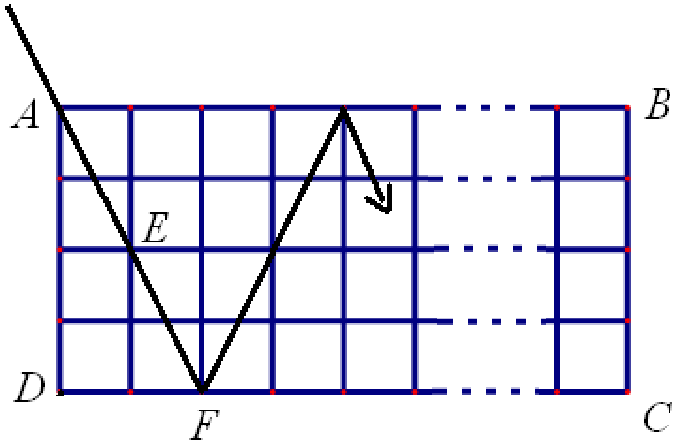
(2) M is divisible by 3: Then the sum of all the digits of M is $a + 2 + 0 + 1 + 5 + b = a + 8 + 6 = a + 14$ which is divisible by 3.
So, $a = 1$, 4, or 7. ③ Since c^3+3 is divisible by $c+3$ and $c^3 + 3 = (c^3 + 3^3) - 24 = (c + 3)(c^2 - 3c + 9) - 24$, 24 is divisible by $c+3$ as well.
Hence, $c = 1$, 3, 5, 9, or 21. ④

Combining ②③④ , we have $c = 1$, 3, 5, or 9 when $a = 1$ and $b = 6$. In this case, the value of $a+b+c$ is 8, 10, 12, or 16. However, when $a = 4$ and $b = 6$, so $c=1$ and $a+b+c = 11$.

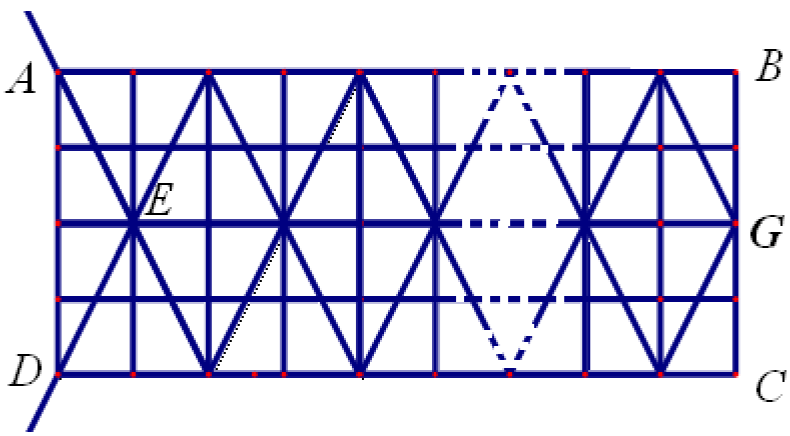
And when $a = 7$ and $b = 6$, $c = 1$ and $a+b+c = 14$.

Therefore, among all the values for $a+b+c$ from above, the largest is $a+b+c = 16$.

T–16. As shown in the figure below, a ray of light enters from point *A* of a $4\times m$ grid graph. This ray will reflect whenever it hits the sides *AB*, *BC*, *CD*, or *AD*. However, if would leave the graph when it hits the corner points *A*, *B*, *C*, or *D*. Suppose this ray of light enters from A and passes through 2016 grid points (including points *A* and *D* and each point would only count once) and then leave the graph at point *D*. Find *m*.



Solution: 1344. As shown in the figure below, because of the property related to the reflection of rays of light, on the side *BC*, it is only possible for this ray to pass through points *B*, *C*, or *G* which is the midpoint of *BC*. However, this ray cannot pass through B and C since it would leave the graph is it passes through those two points.



Hence, this ray will reflect on *G*. Observe the pattern on how this ray passes through the grid points, these grid points must be on *AB*, *CD*, or *EG* and also passes through the same number of grid points on each segment. Since this ray passes through 2016 grid points, it passes through 2016÷3 = 672 grid points on *AB*. Therefore, $m = 672\times 2 = 1344$.

T–17. If positive integers *x* and *y* satisfy the equation $x^3+5x^2y+8xy^2+6y^3=91$, find the value of $x+y$.

Solution: 3.

$$\begin{aligned} x^3+5x^2y+8xy^2+6y^3 &= (x^3+4x^2y+3xy^2)+(x^2y+5xy^2+6y^3) \\ &= x(x^2+4xy+3y^2)+y(x^2+5xy+6y^2) \\ &= x(x+3y)(x+y)+y(x+3y)(x+2y) \\ &= (x+3y)(x^2+2xy+2y^2). \end{aligned}$$

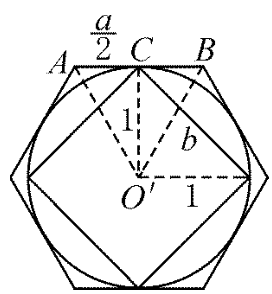
Since x and y are positive integers, so $x+3y>1$ and $x^2+2xy+2y^2>1$. Also, 91=7×13,

so $\begin{cases} x+3y=7, \\ x^2+2xy+2y^2=13, \end{cases}$ ① or $\begin{cases} x^2+2xy+2y^2=7, \\ x+3y=13, \end{cases}$ ②

Since $x^2+2xy+2y^2=13$ and $4^2>13$, $2\times3^2>13$, so $x\le3, y\le2$. Hence, y = 1 or 2.
Substitute y = 1 into $x+3y$ = 7. Then x = 4. This contradicts the fact that $x\le3$.
Substitute y = 2 into $x+3y$ = 7. Then x = 1. Therefore, the solution for is x = 1 and y = 2.
Similarly, use the same reasoning to analyze ②, ② has no solution. Therefore, $x+y$ = 3.

T–18. Suppose a circle of radius 1 that is the inscribed circle of a regular hexagon and also the circumcircle of a square. Let a and b be the edge lengths of the hexagon and square, respectively. If the line $y=-\frac{b}{a}x+\frac{a}{b}$ forms a triangle with x– and y–axes and the inscribed circle of this triangle has a radius of r, find the value of $\frac{1}{r}$.

Solution: $\frac{3+\sqrt{6}+\sqrt{15}}{2}$**.** As shown in the figure below, since a is the edge length of the hexagon, $\angle AO'B$ = 360°÷6 = 60° or $\angle AO'C$ = 60°÷2 = 30°.



Because $\triangle AO'B$ is an equilateral triangle, so $a=\frac{2}{3}\sqrt{3}$.

Also, $b=\sqrt{1^2+1^2}=\sqrt{2}$. Hence, $y=-\frac{b}{a}x+\frac{a}{b}=-\frac{\sqrt{6}}{2}x+\frac{\sqrt{6}}{3}$ and its intersects with the coordinate axes are $M(0, \frac{\sqrt{6}}{3})$ and $N(\frac{2}{3}, 0)$. Therefore,

$$MN=\sqrt{(\frac{\sqrt{6}}{3})^2+(\frac{2}{3})^2}=\frac{\sqrt{10}}{3}.$$

Because $\triangle MON$ is a right triangle and r is the radius of the inscribed circle of this triangle, so $OM\cdot ON=r(OM+ON+MN)$ or

$$\frac{1}{r}=\frac{OM+ON+MN}{OM\cdot ON}=\frac{\frac{\sqrt{6}}{3}+\frac{2}{3}+\frac{\sqrt{10}}{3}}{\frac{\sqrt{6}}{3}\times\frac{2}{3}}=\frac{3+\sqrt{6}+\sqrt{15}}{2}.$$

T–19. Consider the figure below. Given three points $A(-3,0)$, $B\left(\sqrt{3},0\right)$, and $C(0,-3)$. How many possible points $E(x, y)$ (where $0 < y < 4$) that will make ΔABE similar to ΔABC?

Solution: 4. Based on the coordinates of points *A*, *B*, and *C*, it is easy to see that $AB = 3+\sqrt{3}$, $\angle CAB =$ 45°, and $\angle ABC =$ 60°. So, $\angle ACB$ = 75°. To make ΔABE similar to ΔABC, the three interior angles of ΔABE must be also 45°, 60°, and 75°.

(1) As shown in the figure on the right, pick a point E_1 so that $\angle E_1AB$ = 60° and $\angle E_1BA$ = 45°. Drop a perpendicular line from E_1 so it would intersect the *x*–axis at F_1. Then $AF_1 = \frac{\sqrt{3}}{3}E_1F_1$ and $F_1B = E_1F_1$. Since $AF_1 + F_1B = AB$, $\frac{\sqrt{3}}{3}y_1 + y_1 = 3+\sqrt{3}$ or

$y_1 = 3 < 4$.

Similarly, pick a point E_2 so that $\angle E_2AB$=45° and $\angle E_2BA$ = 60°.

Drop a perpendicular line from E_2 and go through the same kind analysis as above, we would find that $y_2 = 3 < 4$.

Therefore, both E_1 and E_2 satisfy our conditions.

(2) As shown in the left figure below, pick a point E_3 so that $\angle E_3AB$ = 75° and $\angle E_3BA$ = 45°. Locate a point *M* on E_3B so that $AM \perp E_3B$ and drop a perpendicular line from E_3 so it would intersect the *x*–axis at F_2. Then

$AM = BM = \frac{\sqrt{2}}{2}AB$, $E_3M = \frac{\sqrt{3}}{3}AM$, and $E_3F_2 = \frac{\sqrt{2}}{2}E_3B$. So,

$AM = BM = \frac{\sqrt{2}}{2}\left(3+\sqrt{3}\right)$, $E_3M = \frac{\sqrt{2}}{2}\left(1+\sqrt{3}\right)$, and

$E_3B = BM + E_3M = \sqrt{2}\left(2+\sqrt{3}\right)$. In this case,

$y = E_3F_2 = 2+\sqrt{3} < 4$.

Similarly, pick a point E_4 so that $\angle E_4AB$ = 45° and $\angle E_4BA$ = 75°. Go through the same kind of analysis as above, we would find that $y = 2+\sqrt{3} < 4$.

Therefore, both E_3 and E_4 satisfy our conditions.

(3) As shown in the right figure below, pick a point E_5 so that $\angle E_5AB = 75°$ and $\angle E_5BA = 60°$. Locate a point N on E_5B so that $AN \perp E_5B$ and drop a perpendicular line from E_5 so it would intersect the x–axis at F_3. Then $BN = \frac{1}{2}AB$, $E_5N = AN = \frac{\sqrt{3}}{2}AB$, and $E_5F_3 = \frac{\sqrt{3}}{2}E_5B$. So, $BN = \frac{1}{2}\left(3+\sqrt{3}\right)$ and $E_5N = AN = \frac{\sqrt{3}}{2}\left(3+\sqrt{3}\right)$. In this case,

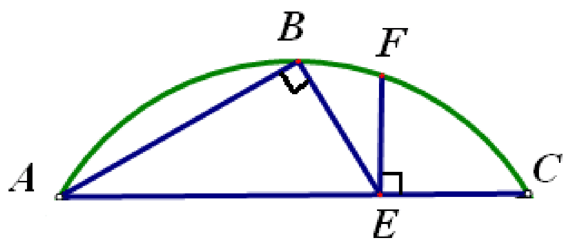
$$y = E_5F_3 = \frac{\sqrt{3}}{2}E_5B = \frac{\sqrt{3}}{2}\left(E_5N + BN\right) = \frac{6+3\sqrt{3}}{2} > 4.$$

Similarly, pick a point E_6 so that $\angle E_6AB = 60°$ and $\angle E_6BA = 75°$. Go through the same kind of analysis as above and we would find that $y = \frac{6+3\sqrt{3}}{2} > 4$.

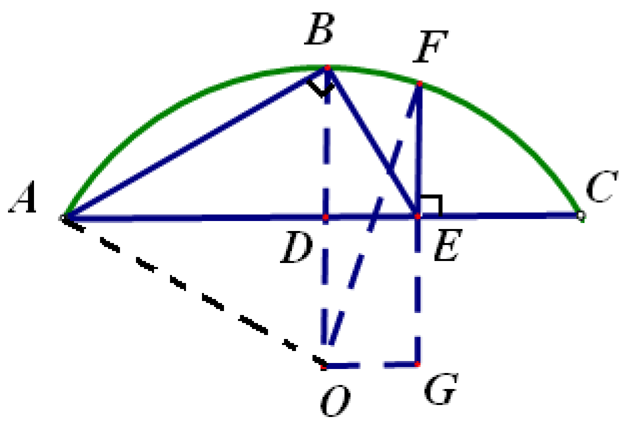
Therefore, E_5 and E_6 do not satisfy our conditions.

(1), (2), and (3) show that there are four possible points E that satisfy $0 < y < 4$.

T–20. As shown in the figure below, point B is the midpoint of arc , point E is on the chord AC, and F is on arc . If arc , , and the radius of the circle for the arc is , find the length of EF.



Solution: $\dfrac{\sqrt{11}-\sqrt{3}}{2}$**.** As shown in the figure below, let O be the center of the circle for $\overset{\frown}{AC}$.



Connect OA, OB, and OF. Let D be the intersection of OB and AC and let G be the point on the extension of FE so that $BO \perp GO$. Because B is the midpoint of arc $\overset{\frown}{AC}$ and arc $\overset{\frown}{AC} = 120^\circ$, so the radius OB is the perpendicular bisector of chord AC and $\angle AOD$=60°. Also, the radius of the circle for arc $\overset{\frown}{AC}$ is $\sqrt{3}$ which means $OA = \sqrt{3}$. Because ΔADO is a 30–60 right triangle with hypotenuse $OA = \sqrt{3}$, so $OD = \dfrac{\sqrt{3}}{2}$ and $AD = \dfrac{3}{2}$. Therefore, $BD = OB - OD = \dfrac{\sqrt{3}}{2}$. Also, since ΔABE is a right triangle,

$$BD^2 = AD \cdot DE \quad \text{or} \quad DE = \frac{BD^2}{AD} = \frac{(\frac{\sqrt{3}}{2})^2}{\frac{3}{2}} = \frac{1}{2}.$$

Now, since quadrilateral DEGO is a rectangle, so $\angle OGF$=90°, OG=DE, and EG=DO.

In right triangle ΔFOG, $FG = \sqrt{OF^2 - OG^2} = \sqrt{OF^2 - DE^2} = \sqrt{(\sqrt{3})^2 - (\frac{1}{2})^2} = \dfrac{\sqrt{11}}{2}$.

Therefore, $EF = FG - EG = FG - OD = \dfrac{\sqrt{11}}{2} - \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{11}-\sqrt{3}}{2}$.