Solutions – Intermediate Division

1. $20 \times 16 = 320$,

hence (A).

2.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{10+5+4}{20} = \frac{19}{20}$$

so that $\frac{1}{20}$ is shaded,

hence (A).

3. Between 11:15 am and 2:09 pm, there are $45 + 120 + 9 = 174$ minutes,

hence (B).

4. (Also J10)
Estimating, $\frac{720163}{2016} \approx \frac{720000}{2000} = \frac{720}{2} = 360$. This suggests that $100 < \frac{720163}{2016} < 1000$.
Checking, $201600 < 720163 < 2016000$ and so $100 < \frac{720163}{2016} < 1000$,

hence (D).

5. $\frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \times \frac{4}{3} = \frac{4}{6} = \frac{2}{3}$,

hence (A).

6. *Alternative 1*

Let the number be x. Then $\frac{3}{4} \times \frac{1}{100} \times x = 6$, so $\frac{x}{400} = 2$ and $x = 800$,

hence (A).

Alternative 2

0.25% of the number is 2 (one-third of the given amount), so 1% of the number is 8 and 100% of the number is 800,

hence (A).

7. (Also J16)
If $A = 1$, then $A \times B + C \times D + E = B + E + C \times D$. On the other hand, if we swap A and E, we get $E \times B + C \times D + A = E \times B + 1 + C \times D$, which is larger, since both B and E are 2 or more. So the largest possible value can't have $A = 1$. Similarly, the largest possible value can't have any of B, C, or D equal to 1. Thus $E = 1$.
Then we only need to consider the following cases.

- $2 \times 3 + 4 \times 5 + 1 = 27$
- $2 \times 4 + 3 \times 5 + 1 = 24$
- $2 \times 5 + 3 \times 4 + 1 = 23$

Therefore, the largest possible value for the expression is 27,

hence (B).

8. (Also S3)
The perimeters of P, Q, R and S are $4\sqrt{2}$, 8, $2\sqrt{2}$ and 4 respectively,

hence (B).

9. *Alternative 1*
The equilateral triangle with corners 12, 4 and 8 has three angles of 60°, so the side from 12 to 8 is 60° from horizontal. The line from 9 to 3 is also horizontal, so the angle between the two lines is 60°,

hence (B).

Alternative 2
The line 1–7 is parallel to 12–8, and so makes the same angle with the line 3–9, due to the corresponding angle rule. Since both 1–7 and 3–9 pass through the centre of the clock, the angle between them is $\frac{2}{12} \times 360° = 60°$,

hence (B).

10. The maximum number of pens I can take without having at least one pen of each colour is $4 + 5 = 9$. This occurs if I take the 4 red pens and the 5 yellow pens. So I need to take 10 pens to be certain that I have at least one pen of each colour,

hence (C).

11. (Also S6)
Alternative 1
The sum of the exterior angles of the pentagon is $360 = 90 + 4 \times (180 - x)$, so that $180 - x = 270/4 = 67.5$ and $x = 180 - 67.5 = 112.5$,

hence (E).

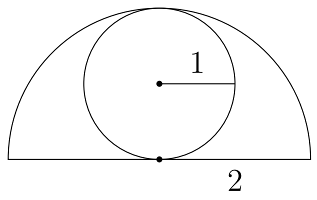
Alternative 2
The sum of the interior angles of the pentagon is $3 \times 180 = 90 + 4x$, so that $x = 450 \div 4 = 112.5$,

hence (E).

12. (Also S7)
If C is the point directly below A and to the left of B, then the right triangle ABC has sides 16 and 12, and hypotenuse x. Then $x^2 = 16^2 + 12^2 = 400$ and $x = 20$,

hence (A).

13. The area of the semicircle is $\frac{1}{2}\pi\, 2^2 = 2\pi$
The area of the circle is π.
Therefore the area not covered is $2\pi - \pi = \pi$,



hence (E).

14. We know $2^{10} = (2^2)^5 = 4^5$, so $4^{n+1} = 4^5$ and $n + 1 = 5$. Then $n = 4$,

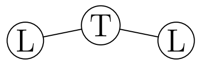
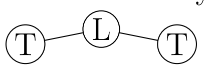
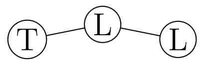
hence (C).

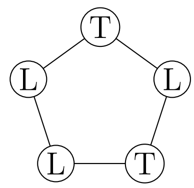
15. (Also UP23, J15)
As the number of points per event is 6 and the total number of points gained is $8 + 11 + 5 = 24$, there must have been 4 events. As Betty has 11 points she must have 3 first places and one second place. Cathy could not have won an event, or her score would be 6 or more, so she must have just one second place. So Adrienne must have two second places,

hence (C).

16. The number $\frac{2016}{N}$ will be the largest square factor of 2016. Factorising into primes, $2016 = 2^5 \times 3^2 \times 7 = 2 \times 7 \times (2^2 \times 3)^2$, so the largest square factor of 2016 is 12^2 and then $N = 14$,

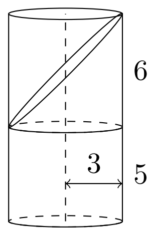
hence (A).

17. Anyone telling the truth is between two liars: 
Anyone lying is not between two liars—they are either between two truth-tellers or one liar and one truth-teller:  or 
So there can't be two truth-tellers in a row nor three liars in a row.
Consequently, at least one person is telling the truth, and is between two liars. The remaining two people can't both be liars nor both truth-tellers, since then there would be four liars or two truth-tellers in a row. So they must be one liar and one truth-teller.
In all, there are three liars and two truth-tellers,



hence (C).

18. The liquid in the glass is equal to a cylinder of height 5 cm and half of a cylinder of height 6 cm.
Thus



$$V = \pi \times 3^2 \times 5 + \frac{1}{2} \times \pi \times 3^2 \times 6 = 45\pi + 27\pi = 72\pi \ \text{cm}^3$$

Since $1\,\text{cm}^3 = 1\,\text{mL}$, the amount of water is 72π mL,

hence (D).

19. (Also S15)
Suppose that m is the average number of correct answers by the seven students whose marks weren't listed. Then we know that m is an integer and $10 \le m \le 20$. The average number of correct answers by all ten students is

$$\frac{8 + 8 + 9 + 7m}{10} = \frac{25 + 7m}{10}$$

So $25 + 7m$ is divisible by 10, which is possible only if m is an odd multiple of 5. However, $10 \le m \le 20$, so that $m = 15$.
Therefore, the average number of correct answers by all ten students is

$$\frac{25 + 7 \times 15}{10} = 13$$

hence (D).

20. In 100 revolutions of the pedals, the length of the first chain that passes over the gears is $100 \times 30 = 3000$ links, and so the 15-tooth gear revolves $3000 \div 15 = 200$ times.

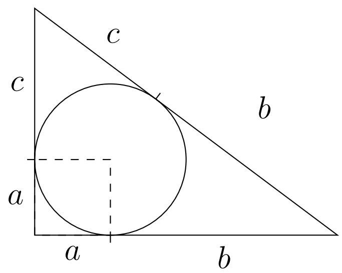
Then $200 \times 32 = 6400$ links of the second chain pass over the gears, and so the pump gear rotates $6400 \div 40 = 160$ times,

hence (A).

21. *Alternative 1*

Two tangents from a point to a circle have equal lengths, so

$$\begin{aligned} a + b &= 8 \\ a + c &= 6 \\ b + c &= \sqrt{6^2 + 8^2} = 10 \\ \\ 2a + 2b + 2c &= 24 \\ a + b + c &= 12 \end{aligned}$$

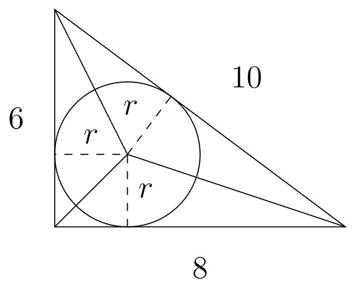


Then $a = 2$, $b = 6$ and $c = 4$. Due to the right angle, the radius of the circle is 2,

hence (B).

Alternative 2

The triangle has sides 6, 8 and 10 and area $\frac{1}{2} \times 6 \times 8 = 24$. It can be cut into three triangles of areas $4r$, $3r$ and $5r$ as shown.



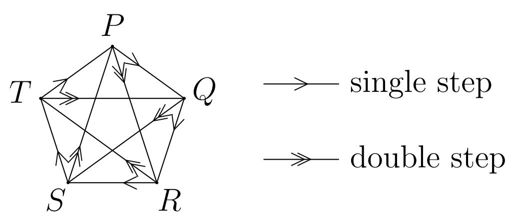
Then $12r = 24$ and $r = 2$,

hence (B).

22. Here is the sequence:

Q			Q						Q

Moving from one letter to the next can only be done as in the diagram.



Counting the single steps as +1 and the double steps as +2, any circuit must add to a multiple of 5.

So to get from Q to Q in three steps (for the first 4 letters in the sequence) requires one single step and two double steps to add to 5.

There are three ways of doing this, $1 + 2 + 2$, $2 + 1 + 2$ and $2 + 2 + 1$, which are the triangles QRT, QST, QSP.

To get from Q to Q in 6 steps (for the remaining letters) the multiple of 5 must be 10, as the total is at least 6 and at most 12. Then there must be 2 steps that are +1 and 4 steps that are +2. These can be listed:

112222	121222	122122	122212	122221
211222	212122	212212	212221	221122
221212	221221	222112	222121	222211

So there are 15 ways of completing the second part of the sequence.
Alternatively, the combinatoric formula $\binom{6}{2} = 15$ can be used.
In all, the number of possible sequences is $3 \times 15 = 45$,

hence (D).

23. Since they arrived home 10 minutes earlier than usual, this has taken 5 minutes off each direction travelled by Alan. So Alan must have met Cynthia at 5:25 pm instead of 5:30 pm. As Cynthia started walking at 5:00 pm, she has walked for 25 minutes,

hence (C).

24. (Also UP29, J27)
There are three cases for how the triangles are coloured:

(i) All three sides are the same colour, with 5 possibilities.
(ii) Two sides are the same and one side is different, with $5 \times 4 = 20$ possibilities.
(iii) All three sides are different, with $\dfrac{5 \times 4 \times 3}{6} = 10$ possibilities.

So there are $5 + 20 + 10 = 35$ possibilities in all,

hence (A).

Note: The calculation in (iii) relies on the observation that if we are choosing the sides in order, there are 5 possibilities for the first side, then 4 for the second, then 3 for the third. However, in these $5 \times 4 \times 3 = 60$ possibilities, each selection xyz will appear 6 times: $xyz, xzy, yxz, yzx, zxy, zyx$. This idea appears in the general formula for $\binom{n}{m}$, the number of ways of choosing m objects from n objects.

25. A super-Fibonacci sequence is determined entirely by its first two terms, 1 and x, say. The sequence then proceeds as follows:

$$1,\ x,\ (1+x),\ 2(1+x),\ 4(1+x),\ 8(1+x),\ \ldots$$

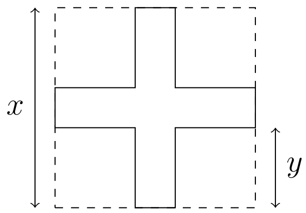
Since adding all previous terms amounts to doubling the last term, all terms from the third onwards are of the form $2^k(1+x)$, for some $k \geq 0$. If $2016 = 2^5 \times 63$ were one of these terms, then k will be one of the 6 values $k = 0, \ldots, 5$, giving 6 possible values for x. These are $x = 62, 125, 251, 503, 1007, 2015$. There is also the possibility that $x = 2016$, so there are 7 such sequences in total,

hence (D).

26. Each of the seven smaller triangles has area $a = \dfrac{2016}{7} = 288\,\text{cm}^2 = 2 \times 12^2$. The right isosceles triangle $\triangle BIE$ is made up of 4 of these, so its area is $4a = 8 \times 12^2 = \frac{1}{2} \times 48^2$ and its two equal sides are $BE = BI = 48\,\text{cm}$.
The area of $\triangle BHD$ is $\dfrac{2}{3}$ the area of $\triangle BHE$, so that $BD = \dfrac{2}{3}BE = 32\,\text{cm}$.
Then in $\triangle BDG$, let $x = BG$, then the area is $288 = \dfrac{1}{2} \times 32x$, so that $x = \dfrac{288}{16} = 18\,\text{cm}$,

hence (18).

27. The cross can be thought of as a large square with four equal small squares removed from its corners. Let the large square have side x and the smaller squares side y.



$$\begin{aligned}\text{Area} &= x^2 - 4y^2 = (x+2y)(x-2y) = 2016 \\ \text{Perimeter} &= 4x\end{aligned}$$

So $x + 2y$ and $x - 2y$ are factors of 2016 that differ by $4y$.

We want x to be as small as possible. This will occur when y is as small as possible, so the two factors of 2016 are as close together as possible.

The closest two are 42 and 48, but this does not give an integer value for y. Next best is 36 and 56, which gives $x + 2y = 56, x - 2y = 36$, so that $2x = 92$ and the perimeter is 184,

hence (184).

28. Because the mean of the 10 scores increased by 0.5 after the mistake was corrected, their sum increased by 5. No other score changed, so Malcolm's score increased by 5. The median of the 10 scores would not have changed if Malcolm's revised score had been 89 or lower or if his original score had been 92 or higher, so his original score must have been between 85 and 91, inclusive.

The possibilities can be summarised as follows:

original score	revised score	original median	revised median
85	90	89.5	90
86, 87, 88, 89	91, 92, 93, 94	89.5	more than 90
90	95	90	91
91	96	90.5	91

Only in the first and last cases does the median increase by 0.5. The sum of Malcolm's two possible correct scores is $90 + 96 = 186$,

hence (186).

29. (Also S26)

Let the formation have r rows and c columns, so the size of the band is rc.

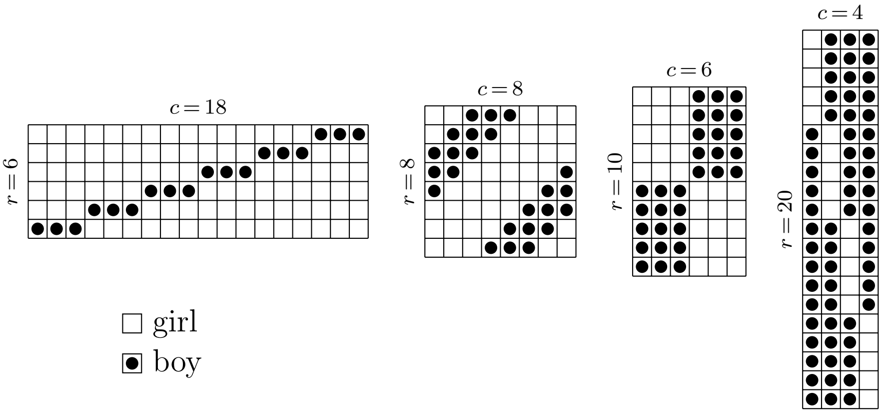
The band contains $3r$ boys and $5c$ girls, so the size of the band is also $3r + 5c$.

Then $rc - 3r - 5c = 0$, or equivalently, $(r - 5)(c - 3) = 15$.

The positive integer solutions for the ordered pair $(r - 5, c - 3)$ are $(1, 15)$, $(3, 5)$, $(5, 3)$, and $(15, 1)$. There are negative integer factorisations of 15, but these have $r \leq 0$ or $c \leq 0$.

The corresponding solutions for (r, c) are $(6, 18)$, $(8, 8)$, $(10, 6)$, and $(20, 4)$, and the corresponding values of rc are 108, 64, 60, and 80.

For each of these sizes, there is at least one arrangement of boys and girls:

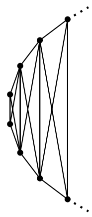


Therefore the sum of all possible band sizes is $108 + 64 + 60 + 80 = 312$,

hence (312).

30. (Also S29)

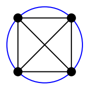
Alternative 1



Draw the 64-gon and all 30 diagonals parallel to a fixed side, dividing it into 31 trapeziums. In each trapezium, draw both diagonals. This requires $64 + 30 + 2 \times 31 = 156$ chords. So the maximum number of chords is 156 or more.

In fact, the maximum is 156, as is shown below.

Firstly, for n points on a circle, where n is even, the same argument tells us that if M_n is the maximum number of chords, then $M_n \geq n + \frac{n}{2} - 2 + 2 \times (\frac{n}{2} - 1) = \frac{5}{2}n - 4$.



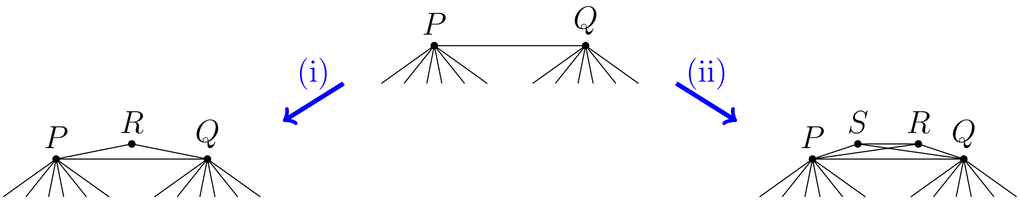
Claim: For all even values of n, $M_n = \frac{5}{2}n - 4$.

Clearly when $n = 4$, $M_n = 6 = \frac{5}{2} \times 4 - 4$, so the claim is true.

For a diagram with a chord PQ on the boundary, consider the following two possible steps:

(i) Between P and Q, add a point R and two chords PR and QR.

(ii) Between P and Q, add two points R, S and five chords PR, PS, QR, QS, RS.

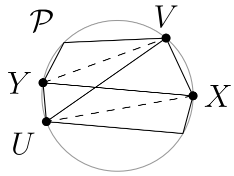


Step (ii) gives more chords per point, so a construction that starts with the 4-point diagram above and builds up using steps (i) and (ii) will have the greatest number of chords if it uses step (ii) as much as possible. For an even number of points, it will only use step (ii). Such a diagram will have $6 + 5 \times \frac{1}{2}(n - 4) = \frac{5}{2}n - 4$ chords.

The only uncertainty is whether a diagram with the maximum number of chords can be built from the $n = 4$ diagram using only steps (i) and (ii).

Suppose the maximum number of chords are drawn. Any chord in the diagram is either an edge of the outer n-gon, a crossed chord or an uncrossed chord. The uncrossed chords divide the n-gon into smaller polygons. If these were triangles and quadrilaterals then the n-gon could be built up using steps (i) and (ii), adding one polygon at a time.

Consider such a polygon $\mathcal{P}$ with each side an uncrossed chord and with 5 or more sides. Some diagonal XY must be a chord in the diagram, since the diagram has the maximum number of chords. Then XY must be crossed by another chord UV, or else the uncrossed chord XY would have split $\mathcal{P}$ into smaller polygons. Any chord passing through any edge of quadrilateral $XUYV$ would cross XY or UV, which would then be double-crossed, which is forbidden. So every edge of $XUYV$ crosses no chords, so it must be in the diagram (due to maximality) where it will be an uncrossed chord.



Now, since $\mathcal{P}$ has 5 or more sides, $XUYV \neq \mathcal{P}$ so at least one side of $XUYV$, say XU, is not a side of $\mathcal{P}$. Then XU is an uncrossed chord that splits $\mathcal{P}$ into smaller polygons, which cannot happen. Hence such a polygon $\mathcal{P}$ must have 4 or fewer edges.

In conclusion, any diagram with the maximum number of chords can be built from the 4-point diagram using steps (i) and (ii). When n is even, the most chords are obtained using only step (ii), which gives $M_n = \frac{5}{2}n - 4$. Consequently $M_{64} = 160 - 4 = 156$,

hence (156).

Alternative 2

As in the first solution, 156 chords are possible.

To see that this is the most, we first note a well-known result, known as *triangulation of a polygon*:

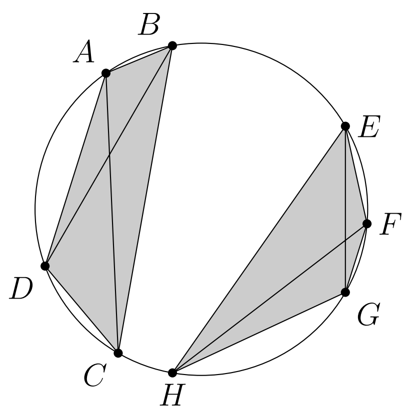
> When a polygon with n sides (n-gon) is cut into triangles, where each triangle's vertices are vertices of the original n-gon, there are $n-2$ triangles. Also, there are $n-3$ cuts, each along a diagonal of the n-gon.

A first consequence of triangulation is that the maximum number of non-intersecting diagonals that can be drawn inside an n-gon is $n-3$.

Secondly, for triangles whose vertices are vertices of the original n-gon, the maximum number of non-overlapping triangles that can be drawn inside the n-gon is $n-2$. This is because we can add more triangles to get a triangulation.

Thirdly, for quadrilaterals whose vertices are vertices of the original n-gon, the maximum number of non-overlapping quadrilaterals that can be drawn inside the n-gon is $\dfrac{n-2}{2} = \dfrac{n}{2} - 1$. This is because each quadrilateral can be split into two non-overlapping triangles.

Returning to the question, for every pair of crossing chords AC and BD, shade in the quadrilateral $ABCD$. For two shaded quadrilaterals $ABCD$ and $EFGH$, neither diagonal AC or BD intersects EG or FH, so $ABCD$ and $EFGH$ do not overlap.



That is, the shaded quadrilaterals are non-overlapping, and so there are at most $\frac{n}{2} - 1 = 31$ of them. Thus there are at most 31 pairs of crossing chords.

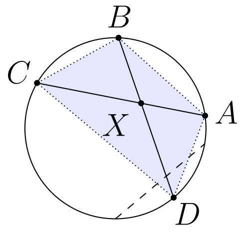
For each pair of crossing chords, remove one chord. There are at most 31 removed chords. The chords remaining have no crossings, and are either sides of the 64-gon (at most 64 of these) or diagonals of the 64-gon (at most 61 of these). Consequently the number of chords originally was at most $64 + 61 + 31 = 156$,

hence (156).

Alternative 3

As in the first solution, 156 chords can be drawn.

To see that 156 is the maximum, suppose the maximum number of chords are drawn—no more can be added. In particular, any possible chord that intersects no other drawn chord must be drawn. This includes all edges of the regular 64-gon. It also means that the only polygons that have all vertices on the circle and no chords inside are triangles, since otherwise a diagonal could be drawn.



When two of the chords AC and BD intersect at an interior point X, there are no other chords intersecting AC and BD, so no other chord will pass inside the quadrilateral $ABCD$. Consequently, the chords AB, BC, CD and DA do not intersect any other chords, so they must be included. Then $ABCD$ appears in the diagram as a *crossed quadrilateral*: all sides and both diagonals are drawn.

So that we can use Euler's formula $f + v = e + 2$, we consider the figure as a planar graph where the vertices include the 64 original points and the intersection points, and the edges include the chords that aren't cut by another chord and the two parts of the chords that are cut by another chord.

There are three types of faces: the exterior of the 64-gon, triangles that are part of a crossed quadrilateral, and triangles that have all vertices on the circle. Suppose there are q crossed quadrilaterals and t triangles. Then

- The number of vertices is $v = 64 + q$, the 64 initial vertices plus one for each crossed quadrilateral.
- The number of faces is $f = 1 + t + 4q$, the outside of the 64-gon, the t triangles, and 4 triangles for each crossed quadrilateral.
- The number of vertices is e where $2e = 64 + 3\,(t + 4q)$. This total is from adding the number of edges on each face, which counts each edge twice.
- The number of chords is $c = e - 2q = 32 + \frac{3}{2}\,t + 4q$, since for each crossed quadrilateral the number of edges is 2 more than the number of chords.

Then in Euler's formula $f + v = e + 2$:

$$
\begin{aligned}
0 = (f + v) - (e + 2) &= (65 + t + 5q) - (34 + \frac{3}{2}\,t + 6q) \\
&= 31 - \frac{1}{2}\,t - q \\
c &= 32 + \frac{3}{2}t + 4(31 - \frac{1}{2}t) \\
&= 156 - \frac{1}{2}\,t
\end{aligned}
$$

Hence the maximum number of chords drawn is 156, attained when $q = 31$ and $t = 0$, hence (156).