Solutions – Junior Division

1. $2016 \times 2 = 4032$,

hence (E).

2. The angles in the triangle add to $180°$, so $x = 180 - 20 - 20 = 140$,

hence (D).

3. 30 days is 4 weeks and 2 days. So 30 days from today is the same as 2 days, which is a Saturday,

hence (E).

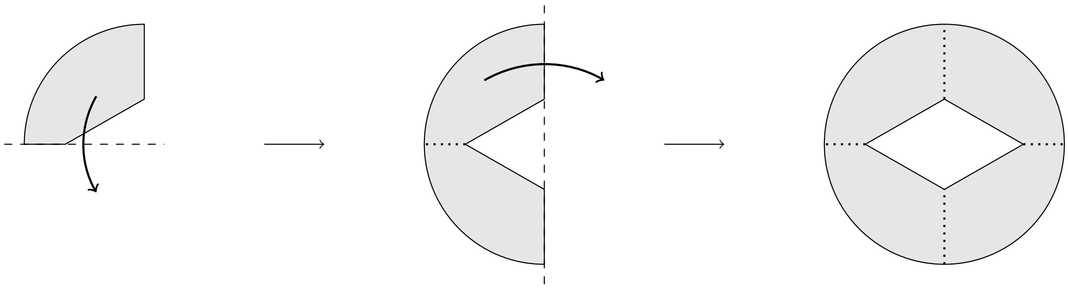
4. $-5 + 8 = 3$,

hence (E).

5. 25% is $\frac{1}{4}$, so 25% of $\frac{1}{2}$ is $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$,

hence (A).

6. (Also MP12)
Each time the paper is unfolded, the two parts will be reflections of each other through the fold line.



hence (A).

7. $100 - (29 + 16 + 8.95) = 100 - 53.95 = 46.05$,

hence (C).

8. In order, $0.009 < 0.019 < 0.08 < 0.109 < 0.4 < 0.409 < 0.91$,

hence (C).

9. 12 noon is 10 minutes after starting, 1 pm is 70 minutes, so 1:04 pm is 74 minutes,
hence (D).

10. (Also I4)
Estimating, $\frac{720163}{2016} \approx \frac{720000}{2000} = \frac{720}{2} = 360$. This suggests that $100 < \frac{720163}{2016} < 1000$.
Checking, $201600 < 720163 < 2016000$ and so $100 < \frac{720163}{2016} < 1000$,

hence (D).

11. The areas of the three squares, from smallest to largest, are 9, 16, and 25 square units. The shaded region has area $16 - 9 = 7$, so the portion of the largest square that is shaded is $100 \times \dfrac{7}{25} = 28$ percent,

hence (B).

12. *Alternative 1*

If Liana has m marbles, Jan has $3m$ marbles and $3m - 3 = m + 3$. Solving this, $2m = 6$ and $m = 3$. Between them, they have $4m = 12$ marbles,

hence (D).

Alternative 2

Jan has $\frac{3}{4}$ of the marbles and Liana has $\frac{1}{4}$. After transferring they each have $\frac{1}{2}$ of the marbles. So the 3 marbles transferred make up $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ of the marbles. Therefore they have 12 marbles between them,

hence (D).

13. In every two rows, there are 7 pavers, one of which will be cut in half. So the number of cut pavers is $1750 \div 7 = 250$,

hence (A).

14. On Monday, I planted 10 apple trees. On Tuesday, the smallest number of orange trees I could have planted is 9, one between each pair of neighbouring apple trees. On Wednesday, the smallest number of peach trees I could have planted is 18, one between each pair of neighbouring apple and orange trees. So the smallest number of trees that I planted altogether is $10 + 9 + 18 = 37$,

hence (C).

15. (Also UP23, I15)

As the number of points per event is 6 and the total number of points gained is $8 + 11 + 5 = 24$, there must have been 4 events. As Betty has 11 points she must have 3 first places and one second place. Cathy could not have won an event, or her score would be 6 or more, so she must have just one second place. So Adrienne must have two second places,

hence (C).

16. (Also I7)

If $A = 1$, then $A \times B + C \times D + E = B + E + C \times D$. On the other hand, if we swap A and E, we get $E \times B + C \times D + A = E \times B + 1 + C \times D$, which is larger, since both B and E are 2 or more. So the largest possible value can't have $A = 1$. Similarly, the largest possible value can't have any of B, C, or D equal to 1. Thus $E = 1$.

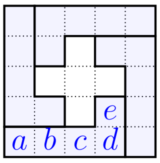
Then we only need to consider the following cases.

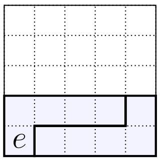
- $2 \times 3 + 4 \times 5 + 1 = 27$
- $2 \times 4 + 3 \times 5 + 1 = 24$
- $2 \times 5 + 3 \times 4 + 1 = 23$

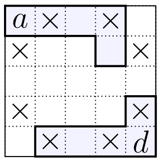
Therefore, the largest possible value for the expression is 27,

hence (B).

17. By trial and error, the pattern in diagram (i) can be found, so that one possibility is that the other tile is (E).

(i)


(ii)


(iii)


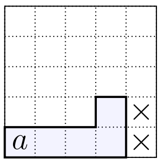
We need to confirm that this is the only possibility.

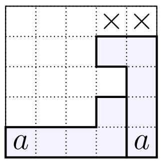
The grid can have at most one of its 4 corners occupied by the other tile, so either 3 or 4 of the L tiles will occupy a corner. Label the squares of each L a, b, c, d, e as shown in diagram (i), then the only squares of an L that can be in the corner of the 5×5 grid are a, d and e.

If e is in a corner, another L must fit in as in diagram (ii). For the remaining 5×3 rectangle to contain two Ls and leave the remaining area in one piece, the two Ls must make another 5×2 rectangle, leaving a 5×1 straight tile. This is not an option, so none of the Ls have square e on a corner.

So only a and d can be in a corner. Consider the 8 squares marked $\times$ in diagram (iii). An L with a in the corner will cover 2 of these, and an L with d in the corner will cover 3. Hence there can't be three Ls with d in the corner, so there must be at least one L with a in the corner.

Place an L with a in a corner as in diagram (iv), then the squares marked $\times$ cannot be filled by any of the tiles (A)–(E), so they only be filled by another L with a in the corner, as in diagram (v).

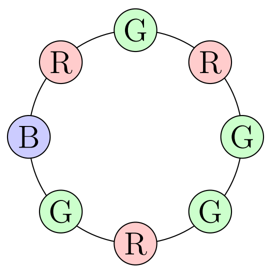
(iv)


(v)


Continuing like this leads to the solution already observed. So there is only one solution,

hence (E).

18. (Also UP20)



First try not to use a blue counter at all.

The counters can't be all red or all green, so start with a red counter at the bottom of the circle. By the first rule, the counters either side must be green.

Then, by the second rule, the counters opposite these green counters must be red. The top counter can now be coloured green.

The side counters cannot be red because they are adjacent to a red. However, only one of them can be green, so the other must be blue. This arrangement, as shown, satisfies the three rules.

Hence, the minimum number of blue counters is 1,

hence (B).

19. For every 3 strands in the original packet, there will be 5 pieces after breakage. Of these 5, 3 are guaranteed to be at least half a strand. So $\frac{3}{5}$ of the pieces are guaranteed to be at least as long as half an unbroken strand,

hence (B).

20. (Also MP30)

Consider the prime factorisation of $2016 = 2^5 \times 3^2 \times 7$. Factors of 2016 under 10 are 1, 2, 3, 4, 6, 7, 8 and 9.

Only 7 has prime factor 7, so this must be one of the ages.

The 3^2 in the prime factorisation will either be from $3 \times 6 = 2 \times 3^2$ or from $9 = 3^2$.

In the first case, the factorisation is $3 \times 6 \times 7 \times 16$, where 16 is too large.

In the second case, two of the ages are 7 and 9. Then the remaining two ages multiply to $2^5 = 32$, so they must be 4 and 8.

Hence the ages are 4, 7, 8 and 9, which add to 28,

hence (C).

21. *Alternative 1*

The large jug is between $50/4 = 12\frac{1}{2}$ L and $50/3 = 16\frac{2}{3}$ L. So its capacity x is either 13, 14, 15, or 16 litres. The amount left in the barrel is either 11, 8, 5 or 2 litres. Call this quantity $y = 50 - 3x$.

The small jug has capacity z between $y/4$ and $y/3$. The options are shown in the table:

x	y	$\frac{y}{4}$	$\frac{y}{3}$	possible z
13	11	$2\frac{3}{4}$	$3\frac{2}{3}$	3
14	8	2	$2\frac{2}{3}$	—
15	5	$1\frac{1}{4}$	$1\frac{2}{3}$	—
16	2	$\frac{1}{2}$	$\frac{2}{3}$	—

So the small jug has capacity 3 litres,

hence (C).

Alternative 2

Suppose one large jug holds x litres and one small jug holds y litres. If $x = 17$ or more, then 3 large jugs cannot be filled, and if $x = 12$ or less then more than 3 large jugs would be filled. So $x = 13, 14, 15$ or 16.

If $x = 16$, then 2 litres remain, and 3 small jugs can't be filled.

If $x = 15$, then 5 litres remain, and the small jug must hold 1 litre. But then 5 small jugs can be filled.

If $x = 14$, then 8 litres remain, and the small jug must hold 2 litres. But then 4 jugs can be filled.

Finally, if $x = 13$, then 11 litres remain. Then since $11 = 3 \times 3 + 2$, each small jug must hold 3 litres,

hence (C).

22. *Alternative 1*

If the middle digit is 1, then the number must either have the form $\boxed{2}\boxed{}\boxed{1}\boxed{}\boxed{}$ or $\boxed{}\boxed{}\boxed{1}\boxed{}\boxed{2}$. In each case the 4 cannot be adjacent to either the 3 or the 5, so it must go between the 1 and the 2, and there are then 2 ways to place the 3 and the 5. Therefore there are 4 such numbers whose middle digit is 1. By symmetry there are also 4 such numbers whose middle digit is 5.

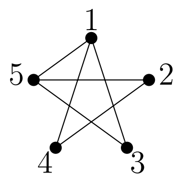
If the middle digit is 2, then the number must either have the form $\boxed{1}\boxed{}\boxed{2}\boxed{}\boxed{3}$ or $\boxed{3}\boxed{}\boxed{2}\boxed{}\boxed{1}$. In each case the 4 cannot be adjacent to the 3, so it must be between the 1 and the 2, and the position of the 5 is then determined. Therefore there are 2 such numbers whose middle digit is 2. By symmetry there are also 2 such numbers whose middle digit is 4.

If the middle digit is 3, then the number must either have the form $\boxed{2}\boxed{}\boxed{3}\boxed{}\boxed{4}$ or $\boxed{4}\boxed{}\boxed{3}\boxed{}\boxed{2}$. In each case the 1 must be between the 3 and the 4, and the position of the 5 is then determined. Therefore there are 2 such numbers whose middle digit is 3.

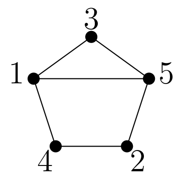
The total number of numbers with the required property is $4 + 4 + 2 + 2 + 2 = 14$, hence (B).

Alternative 2

In the first diagram, two digits are joined if they can be neighbouring digits in the number. The 5-digit numbers in the question correspond to paths that visit every digit exactly once.



The second diagram has the same edges, but is rearranged for clarity.



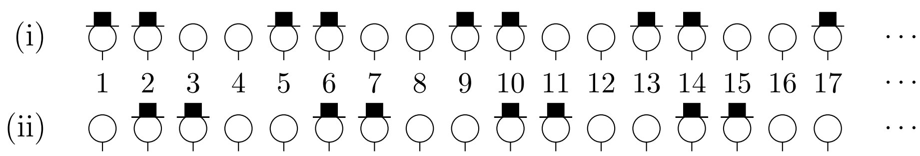
If edge 1–5 is not used, there are 10 possibilities, since there are 5 choices of starting digit, then 2 choices of second digit, and then all other digits follow.

If edge 1–5 is used, then there are 4 possibilities. These can be counted by choosing either edge 1–3 or edge 5–3, and then deciding whether the path will start or end at 3, since the rest of the path is determined by this. This gives four possibilities: 31524, 42513, 35142 and 24153.

In all there are $10 + 4 = 14$ possibilities,

hence (B).

23. Number the people $1, 2, 3, \ldots$. Person 1 either (i) does wear a hat or (ii) does not. In either case, for persons 2, 3, 4, ..., we decide whether they have a hat by making sure that persons 1, 2, 3, ... have exactly one neighbour with a hat:



Each of these patterns repeats every 4 people, since if n is hatless then $n+2$ is hatted, and vice-versa.

From these patterns, persons $1, 3, 5, 7, \ldots$ (odd numbers) may or may not have hats, persons $4, 8, 12, 16, \ldots$ (multiples of 4) never have hats and persons $2, 6, 10, 14, \ldots$ (even, not multiples of 4) always have hats.

Just as the second person must have a hat, so must the second-last. However, person 100 can't have a hat, so there can't be 101 people. The other answers 98, 99, 100 and 102 are all possible, since none of $97, 98, 99$ and 101 is a multiple of 4, so each will have a hat in either (i) or (ii) above,

hence (D).

24. *Alternative 1*

At each step, the person receiving lollies doubles their pile. Working backwards,

Josh	Ruth	Sam	
32	32	32	(Lollies at end)
16	16	64	(Josh and Ruth about to double to 32)
8	56	32	(Josh about to double to 16 and Sam to 64)
52	28	16	(Ruth about to double to 56 and Sam to 32)

hence (E).

Alternative 2

Suppose Josh starts with x lollies out of the 96 total.

Ruth and Sam together have $(96 - x)$, so Josh gives away $(96 - x)$ lollies, leaving $x - (96 - x) = 2x - 96 = 2(x - 48)$.

Ruth then gives Josh $2(x - 48)$ more lollies so that Josh has $4(x - 48)$.

Sam then gives Josh $4(x - 48)$ more lollies so that Josh has $8(x - 48)$ in the end.

Solving, $8(x - 48) = 32$, then $x - 48 = 4$ and $x = 52$,

hence (E).

25. *Alternative 1*

For convenience, represent the lines by symbols a, b, c or d so that lines use the same symbol if and only if they rhyme. Note that structures such as $abac$ and $cdca$ are not considered different since either one indicates that the first and third lines rhyme with each other, while neither of the second or fourth lines rhymes with any others. The following rules will generate a list of different rhyming structures:

1. The first letter must be a.
2. A letter can be used only if all of its predecessors in the alphabet have already been used.

The full list, in alphabetical order, is

$$aaaa, aaab, aaba, aabb, aabc, abaa, abab, abac, abba, abbb, abbc, abca, abcb, abcc, abcd$$

so there are 15 different rhyming structures in total. To count them more systematically, consider the five three-line poems:

$$aaa, aab, aba, abb, abc.$$

All four-line poems are constructed by appending a single letter to these, subject to rule 2 above. The 15 possibilities are

$$\begin{array}{lll} aaa & +a \text{ or } b & = 2 \text{ possibilities} \\ \left.\begin{array}{l} aab \\ aba \\ abb \end{array}\right\} & +a, b \text{ or } c & = 3 \times 3 = 9 \text{ possibilities} \\ abc & +a, b, c \text{ or } d & = 4 \text{ possibilities} \end{array}$$

hence (B).

Alternative 2

Classify the rhyming patterns by the number of lines that rhyme:

(i) With all 4 lines rhyming, there is only one pattern 1

(ii) With 3 lines rhyming and one line not, the unrhymed line is line 1, 2, 3 or 4. So there are 4 patterns 4

(iii) With 2 pairs of 2 lines rhyming, the line that rhymes with line 1 is line 2, 3 or 4. So there are 3 patterns 3

(iv) With 2 lines rhyming and the other two not rhyming, there are 2 patterns for every pattern in (c). So there are 6 patterns 6

(v) If no lines rhyme, there is only 1 pattern 1

Then $1 + 4 + 3 + 6 + 1 = 15$,

hence (B).

26. Let N be the number abc. In the units column, $4c$ has units digit 2, so $c = 3$ or 8. If $c = 8$, then $18000 < 24N < 19000$. From the multiples of 24 listed below, we have $24 \times 700 = 16800$ and $24 \times 800 = 19200$ so that N must be in the 700s and $a = 7$. In the tens column, we must have $3 + 8 + 6 = 17$:

$$
\begin{array}{ccccc}
 & & 7 & b & 8 \\
\times & & & 2 & 4 \\
\hline
 & & & 3 & 2 \\
 & & ? & ? & \\
 & 2 & 8 & & \\
 & & 1 & 6 & \\
 & ? & ? & & \\
1 & 4 & & & \\
\hline
1 & 8 & b & 7 & 2 \\
\hline
\end{array}
$$

n	$24n$	n	$24n$
1	24	6	144
2	48	7	168
3	72	8	192
4	96	9	216
5	120		

Then $4b$ ends in 8, so $b = 2$ or $b = 7$, but neither of these work.

If $c = 3$, then $13000 < 24N < 14000$. From the multiples of 24, $500 < N < 600$, so $a = 5$. Then in the tens column, $1 + 8 + 6 = 15$:

$$
\begin{array}{ccccc}
 & & 5 & b & 3 \\
\times & & & 2 & 4 \\
\hline
 & & & 1 & 2 \\
 & & ? & ? & \\
 & 2 & 0 & & \\
 & & & 6 & \\
 & ? & ? & & \\
1 & 0 & & & \\
\hline
1 & 3 & b & 5 & 2 \\
\hline
\end{array}
$$

Again, $4b$ ends in 8 so that $b = 2$ or $b = 7$. Clearly $b = 2$ is too small, but $b = 7$ gives a solution: $573 \times 24 = 13752$. So $N = 573$ is the only solution,

hence (573).

27. (Also UP29, I24)

There are three cases for how the triangles are coloured:

(i) All three sides are the same colour, with 5 possibilities.

(ii) Two sides are the same and one side is different, with $5 \times 4 = 20$ possibilities.

(iii) All three sides are different, with $\dfrac{5 \times 4 \times 3}{6} = 10$ possibilities.

So there are $5 + 20 + 10 = 35$ possibilities in all,

hence (35).

Note: The calculation in (iii) relies on the observation that if we are choosing the sides in order, there are 5 possibilities for the first side, then 4 for the second, then 3 for the third. However, in these $5 \times 4 \times 3 = 60$ possibilities, each selection xyz will appear 6 times: $xyz, xzy, yxz, yzx, zxy, zyx$. This idea appears in the general formula for $\binom{n}{m}$, the number of ways of choosing m objects from n objects.

28. Let abc be the three-digit number. Then $100a + 10b + c = 37a + 37b + 37c$. This gives $63a = 27b + 36c$ and dividing by 9, $7a = 3b + 4c$. This gives many solutions where $a = b = c$, but it is a requirement that a, b and c are all different. Hence we are looking for $3b + 4c$ to be a multiple of 7 where $b \neq c$.

As we are looking for the largest, we try $a = 9$. Then $3b + 4c = 63$, which has solutions $(1, 15)$, $(5, 12)$, $(9, 9)$, $(13, 6)$, $(17, 3)$ and $(21, 0)$. None of these work.

Trying $a = 8$, then $3b + 4c = 56$, giving $(0, 14)$, $(4, 11)$, $(8, 8)$, $(12, 5)$ and $(16, 2)$. None of these work.

Trying $a = 7$, then $3b + 4c = 49$ giving $(3, 10)$, $(7, 7)$, $(11, 4)$ and $(15, 1)$. None of these work.

Trying $a = 6$, then $3b + 4c = 42$ giving $(2, 9)$, $(6, 6)$, $(10, 3)$ and $(14, 0)$. So the largest solution is 629,

hence (629).

Note: If you know or observe that $3 \times 37 = 111$, this gives an efficient way to search for solutions without the above algebra.

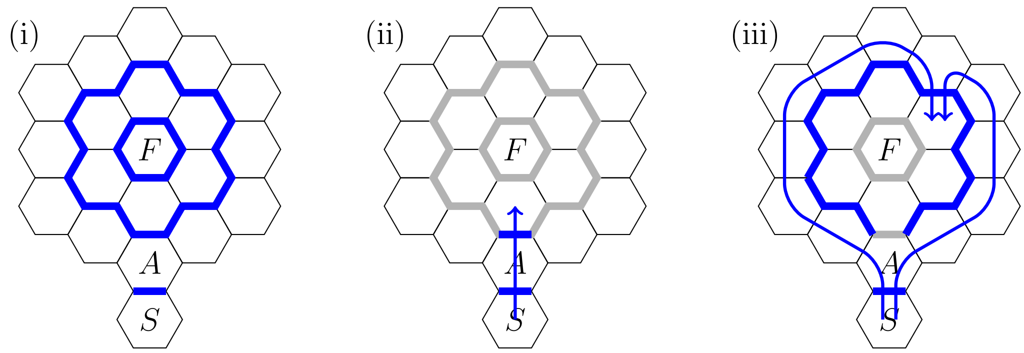
29. Since each term is the sum of the previous two, the amount of error follows the same pattern. Starting from the 89th term which is correct and the 90th term which has an error of 1, we have the following:

Term	89	90	91	92	93	94	95	96	97	98	99	100
Error	0	1	1	2	3	5	8	13	21	34	55	89

Note that we don't actually need to know whether the first error was above or below the correct value,

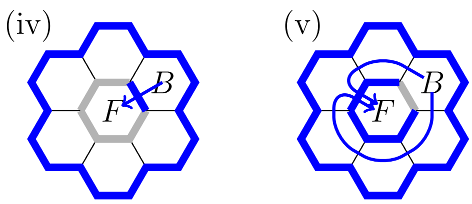
hence (89).

30. Call the levels 0 (paver S) up to 3 (column F). Diagram (i) shows the possible edges where I can step up.



There are 18 edges between level 1 and level 2. For the top edge of column A, diagram (ii) shows the only path from S that finishes by crossing that edge. Diagram (iii) shows that for the other 17 edges there are two paths. So there are 35 paths to level 2 that end by stepping across one of these edges.

Now if B is the level-2 column first stepped on, then of the 6 edges on column F, one has one path from B to F directly crossing that edge, and the other 5 have two paths, as shown in diagrams (iv) and (v) respectively. So there are 11 paths from B to F.



There are no restrictions on combining the path from S to B with a path from B to F, so the total number of paths is $35 \times 11 = 385$,

hence (385).