Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2024

Senior Section (Round 1)

Tuesday, 28 May 2024 **0930 – 1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Find the largest positive integer A such that $2^x + \frac{2025}{2^x} - A > 0$ for all real number x.

(A) 59 (B) 69 (C) 79 (D) 89 (E) 99

2. If $x = \frac{1}{\log_{\frac{2024}{2023}} 7} + \frac{1}{\log_{\frac{2023}{2022}} 7} + \frac{1}{\log_{\frac{2022}{2021}} 7}$, find 7^x.

(A) $\frac{2021}{2024}$ (B) $\frac{2024}{2021}$ (C) $\frac{2022}{2024}$ (D) $\frac{2024}{2022}$ (E) 2024

3. Simplify $\frac{2024}{\sqrt{4+\sqrt{12}}} + \frac{2024}{\sqrt{4-\sqrt{12}}}$.

(A) 1012 (B) $1012\sqrt{3}$ (C) 2024 (D) $2024\sqrt{3}$ (E) $1012 + 1012\sqrt{3}$

4. Suppose $x^{\frac{1}{3}} + 12 = y^{\frac{1}{3}}$ for some real numbers x and y. Find the minimum possible value of $y - x$.

(A) 432 (B) 532 (C) 632 (D) 732 (E) None of the above

5. Find the largest possible value of $\frac{\sqrt{2}\cos(2x)}{\sin(x) + \cos(x)}$.

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) $\frac{5}{2}$

Short Questions

6. If $\sqrt{x+\sqrt{x}} + \sqrt{x-\sqrt{x}} = 4$, find the value of $15x$.

7. Find the smallest positive integer K such that

$$x^2 - 200x + y^2 = 0 \quad \text{and} \quad x + y \leq K.$$

8. Given that $\frac{\cos(x)}{\sin(3x)} - \frac{\sin(x)}{\cos(3x)} - 2 \cdot \frac{\sin(4x)}{\cos(6x)} = 2024$. Find the value of $\frac{\cos(10x)}{\sin(12x)}$.

9. Find the smallest positive integer k such that the coefficient of x^k in the expansion of $\left(5x^3 + \frac{1}{\sqrt{x}}\right)^{2024}$ is **not** zero.

10. Let

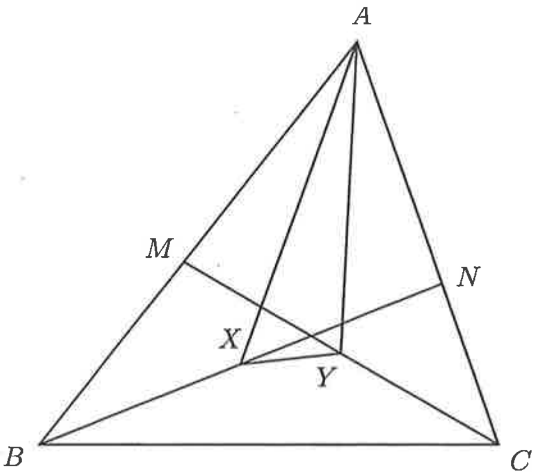
$$P = (2024^2+1)\left(2024^{2^2}+1\right)\left(2024^{2^3}+1\right)\cdots\left(2024^{2^{10}}+1\right) \times 2025 + \frac{1}{2023}.$$

Find the smallest positive integer N such that $N > \log_{2024} P$.

11. Let $\triangle ABC$ be a triangle with area 1000. Let M and N be points on AB and AC respectively such that

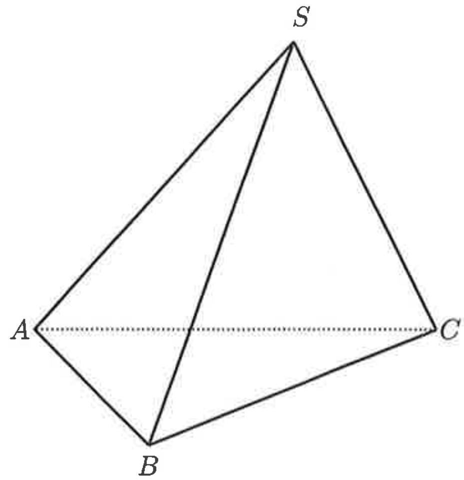
$$AM : MB = 3 : 2 \quad \text{and} \quad AN : NC = 7 : 3.$$

Let X and Y be the midpoints of BN and CM respectively. Find the area of $\triangle AXY$.



12. Find the largest positive integer $n \leq 10000$ such that $1 + 2024n^2$ is a perfect square.

13. In a tetrahedron $SABC$, the faces SBC and ABC are perpendicular to each other. The angles $\angle ASB$, $\angle BSC$, $\angle ASC$ are all $60°$, and $SB = SC = 4$. Find the square of the volume of the tetrahedron.



14. Let a, b, c be the three real roots of the cubic equation

$$2x^3 - 4x^2 - 21x - 8 = 0$$

Given that

$$S = \frac{1}{ab + c - 1} + \frac{1}{bc + a - 1} + \frac{1}{ca + b - 1}$$

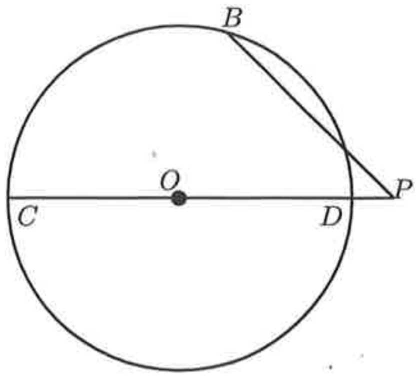
is a rational number that can be expressed as a fraction in the lowest form $\frac{m}{n}$, find the value of $m^2 + n^2$.

15. Consider the equation

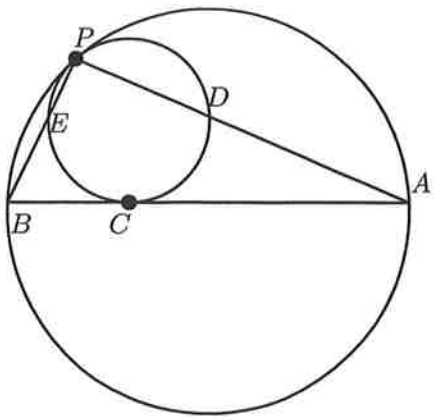
$$\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2}.$$

For the range $0 < x < \pi/2$, the sum of the solutions of the equation can be expressed in the form $\frac{m\pi}{n}$, where $\frac{m}{n}$ is a fraction in the lowest form. Find $m + n$.

16. An engineer constructs a circle with centre O and diamater CD on level ground, and builds a vertical tower of height 20 at the centre. B is another point on the circumference and P is on CD produced such that PB is a secant line of the circle. Given that $PB = 33$, $PC = 77$, and $CD = 74$, find the minimum possible distance of any point on PB to the top of the tower.



17. P is a common point of tangency of two circles. BA is a chord of the larger circle which is tangent to the smaller circle at a point C. PB and PA intersect the smaller circle at points E and D respectively. If $BA = 15$, $PE = 2$, and $PD = 3$, find the length CA.



18. On each face of a cube, an integer greater than 2 is written. Each vertex of the cube is the intersection of three unique faces, and each edge is the intersection of two unique faces. Assign to each vertex the product of the numbers written on the faces intersecting the vertex, and assign to

each edge the product of the numbers written on the faces intersecting the edge. The sum of the numbers assigned to the eight vertices is equal to 2024. Find the maximum possible value of an edge.

19. Find the sum of the squares of each of the roots of the equation

$$x^2 - 4\lfloor x \rfloor - 12 = 0,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

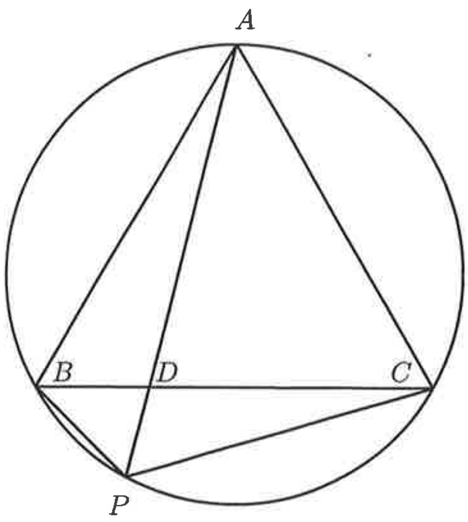
20. Calculate the remainder when 1901^{2024} is divided by 1216.

21. Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial with non-negative integer coefficients satisfying $0 \leq a_i \leq 17$ for all i. If $P(18) = 367616$, find the value of $P(3)$.

22. Evaluate the sum

$$\frac{2}{1+\tan\left(\frac{\pi}{260}\right)} + \frac{2}{1+\tan\left(\frac{2\pi}{260}\right)} + \frac{2}{1+\tan\left(\frac{3\pi}{260}\right)} + \cdots + \frac{2}{1+\tan\left(\frac{129\pi}{260}\right)}.$$

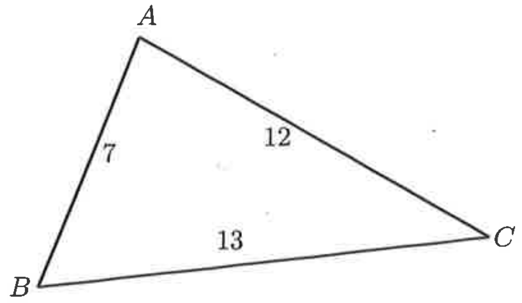
23. An equilateral triangle ABC is inscribed in a circle and P is a point on the minor arc BC. Point D is the intersection of AP and BC.



Suppose that $BP = 5$, $CP = 20$. Find the length of AD.

24. Find the number of positive integers $x < 9000$ such that $x^3 + 95$ is divisible by 96.

25. A scalene triangle $\triangle ABC$ has sides $AB = 7$, $AC = 12$, and $BC = 13$. Write $\tan\frac{A-B}{2}\tan\frac{C}{2}$ as a fraction $\frac{m}{n}$ in its lowest form and find $m + n$.


A
7
12
13
B
C