Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2015
(Open Section, Round 1)

Thursday, 4 June 2015 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

In this paper, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$.

1. Find the largest positive integer N for which $n^5 - 5n^3 + 4n$ is divisible by N for all positive integers n.

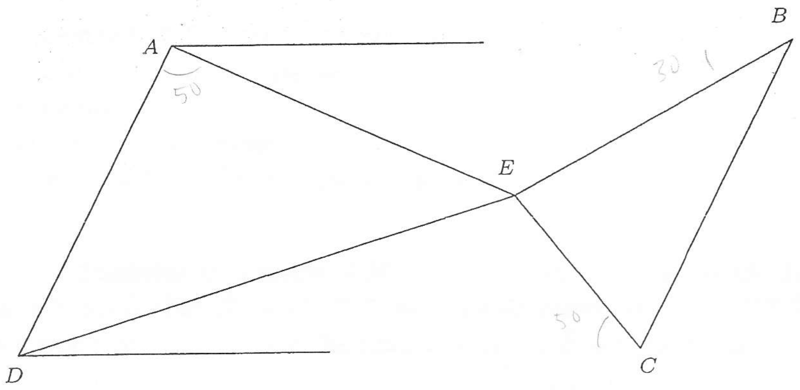
2. Consider all sequences of numbers with distinct terms which follow a geometric progression such that the first, second and fourth terms of the sequence are three consecutive terms of an arithmetic progression. Find the sum of squares of all the possible common ratios of these sequences.

3. Suppose that a given sequence $\{x_n\}$ satisfies the conditions that $x_1 = 1$ and, for $n \geq 1$,

$$x_{n+1} = \frac{1}{16}(1 + 4x_n + \sqrt{1 + 24x_n}).$$

Determine $\lim\limits_{n \to \infty} 3x_n$.

4. In the figure below, E is a point inside the parallelogram $ABCD$ such that $\angle DAE = \angle DCE = 50^\circ$ and $\angle ABE = 30^\circ$. Find $\angle ADE$ in degrees.



5. It is given that $S = \dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \cdots + \dfrac{1}{2014 \times 2015}$. Find the value of $\lfloor 4030 \times S \rfloor$.

6. Let $ABCD$ be a trapezium with AB parallel to DC, and that $AB = 20$ cm, $CD = 30$ cm, $BD = 40$ cm and $AC = 30$ cm. Let E be the intersection of AC and BD. Find the area of triangle DEC in cm^2.

7. How many distinct integers are there in the following sequence:

$$\left\lfloor \frac{1^2}{2015} \right\rfloor, \left\lfloor \frac{2^2}{2015} \right\rfloor, \left\lfloor \frac{3^2}{2015} \right\rfloor, \ldots, \left\lfloor \frac{2015^2}{2015} \right\rfloor?$$

8. Let S be the sum of all the positive solutions of the equation $\sqrt{\dfrac{4 - x^2}{3}} + \sqrt{\dfrac{x^2 - 1}{3}} = 1$. Find $\lfloor S \rfloor$.

9. Given that $\dfrac{(1+10)(1+10^2)(1+10^4)\cdots(1+10^{2^m})}{1+10+10^2+10^3+10^4+\cdots+10^{127}} = 1$, find the value of m.

10. Let a_n be the nth term of a geometric progression, where $a_1 = 1$ and $a_3 = 3$. Find

$$\left(\sum_{i=0}^{10}\binom{10}{i}a_{i+1}\right)\cdot\left(\sum_{j=0}^{10}(-1)^j\binom{10}{j}a_{j+1}\right).$$

11. Let $y = \sqrt{38x-152}+\sqrt{2015-403x}$ be a real function. Find the largest possible value of y.

12. Find the coefficient of x^{50} in the expansion of $(x+1)(x+2)(x+3)\cdots(x+50)(x+51)(x+52)$.

6 digit

13. Fifty numbers from the set $\{1, 2, \ldots, 100\}$ are chosen and another fifty numbers from the set $\{101, 102, \ldots, 200\}$ are chosen. It is known that no two chosen numbers differ by 0 or 100. Determine the sum of all the 100 chosen numbers.

14. Let $H = \left(x^3 - x^2 + x\right)^9$, where $x = \dfrac{2}{\sqrt{5}-1}$. Determine $\lfloor H \rfloor$.

15. Assume that $x \geq y \geq z \geq \frac{\pi}{12}$ and $x+y+z = \frac{\pi}{2}$. Let M and m be the largest possible value and the smallest possible value of $\cos x \sin y \cos z$ respectively. Determine the value of $\left\lfloor \frac{M}{m} \right\rfloor$.

16. Let $\mathrm{f} : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that, for any $x, y \in \mathbb{R}$,

$$(x-y)\mathrm{f}(x+y) - (x+y)\mathrm{f}(x-y) = (6x^2y + 2y^3)(x^2 - y^2).$$

Suppose $\mathrm{f}(1) = -999$. Determine the value of $\mathrm{f}(10)$.

17. Let b, c, d and e be real numbers such that the following equation

$$x^5 - 20x^4 + bx^3 + cx^2 + dx + e = 0$$

has real roots only. Find the largest possible value of b.

18. Let $\mathbb{N}$ denote the set of all positive integers. Suppose that $\mathrm{f} : \mathbb{N} \to \mathbb{N}$ satisfies

(a) $\mathrm{f}(1) = 1$,

(b) $3\mathrm{f}(n)\mathrm{f}(2n+1) = \mathrm{f}(2n)(1 + 3\mathrm{f}(n))$ for all $n \in \mathbb{N}$,

(c) $\mathrm{f}(2n) < 6\mathrm{f}(n)$ for all $n \in \mathbb{N}$.

Determine $\mathrm{f}(2015)$.

19. Let x_1, x_2, x_3, x_4 and x_5 be positive integers such that

$$x_1 + x_2 + x_3 + x_4 + x_5 = x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5.$$

Find the largest possible value of x_5.

20. Ah Meng is going to pick up 2015 peanuts on the ground in several steps according to the following rules. In the first step, he picks up 1 peanut. For each next step, he picks up either the same number of peanuts or twice the number of peanuts of the previous step. What is the minimum number of steps that he can complete the task?

21. Determine the number of integers in the set $S = \{1, 2, 3, \cdots, 10000\}$ which are divisible by exactly one of integers in $\{2, 3, 5, 7\}$.

22. Determine the largest integer n such that

$$\sum_{i=1}^{n} x_i^2 \geq x_n \sum_{i=1}^{n-1} x_i$$

for all real numbers $x_1, x_2, \ldots, x_n$.

23. A circle ω_1 centred at O_1 intersects another circle ω_2 centred at O_2 at two distinct points P and Q. Points A and B are on ω_1 and ω_2 respectively such that AB is an external common tangent to ω_1 and ω_2. The line through PQ intersects the segments AB and O_1O_2 at M and N respectively. Suppose the radius of ω_1 is 143 cm, the radius of ω_2 is 78 cm and $O_1O_2 = 169$ cm. Determine the length of MN in centimetres.

24. Let XY be a diameter of a circle ω of radius 10 cm centered at O. Let A and B be the points on XY such that X, A, O, B, Y are in this order and $AO = OB = 4$ cm. Suppose that P is a point on ω such that the lines PA and PB intersect ω at C and D respectively with C and D distinct from P. Given $\dfrac{PB}{BD} = \dfrac{16}{21}$, determine the ratio $\dfrac{PA}{AC}$.

25. In a triangle ABC, the incircle ω centred at I touches the sides BC, CA and AB at D, E and F respectively, Q is the point on ω diametrically opposite to D, and P is the intersection of the lines FQ and DE. Suppose that $BC = 50$ cm, $CA = 49$ cm and $PQ = QF$. Determine the length of AB in centimetres.