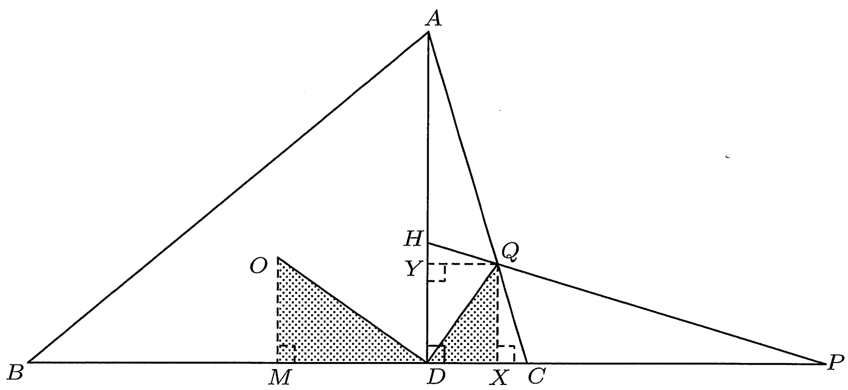
# Singapore Mathematical Society

## Singapore Mathematical Olympiad (SMO) 2011

### (Open Section, Round 2 solutions)

**1.** Drop perpendiculars $OM$ and $QX$ onto $BC$, and $QY$ from $Q$ onto $AD$. First $2DM = DM + BD - BM = BD - (BM - DM) = PD - (CM - DM) = PD - CD = PC$. It is a well-known fact that $2OM = AH$.




Next $\angle CPQ = \angle DBH = \angle HAQ$ so that the triangles $CPQ$ and $HAQ$ are similar. Thus the triangles $XPQ$ and $YAQ$ are similar. Therefore

$$\frac{QX}{DX} = \frac{QX}{QY} = \frac{PC}{AH} = \frac{DM}{OM}.$$

Hence the triangles $DXQ$ and $OMD$ are similar. It follows that $\angle ODQ = 90°$.

**2.** Suppose that at most 2 squares are colored red in any $2 \times 2$ square. Then in any $9 \times 2$ block, there are at most 10 red squares. Moreover, if there are 10 red squares, then there must be 5 in each row. This can be seen as follows. There are 8 $2 \times 2$ blocks. Counting multiplicity, there are altogether 16 red squares. Each red square in the interior is counted twice while each red square at the edge is counted once. If there are 11 red squares, then there are at least 7 red squres in the interior. Thus the total count is at least $4 + 7 \times 2 = 18 > 16$, a contradiction. If there are exactly 10 red squares, then 4 of them must be at the edge and the red squares in each row are not next to each other and hence there 5 in each row.

Now let the number of red squares in row $i$ be $r_i$. Then $r_i + r_{i+1} \leq 10$, $1 \leq i \leq 8$. Suppose that some $r_i \leq 5$ with $i$ odd. Then

$$(r_1 + r_2) + \cdots + (r_{i-2} + r_{i-1}) + r_i + \cdots + (r_8 + r_9) \leq 4 \times 10 + 5 = 45$$

which leads to a contradiction. On the other hand, suppose that $r_1, r_3, r_5, r_7, r_9 \geq 6$. Then the sum of any 2 consecutive $r_i$'s is $\leq 9$. Again we get a contradiction as

$$(r_1 + r_2) + \cdots + (r_7 + r_8) + r_9 \leq 4 \times 9 + 9 = 45.$$

**3.** Let $r = 1/x, s = 1/y, t = 1/z$. There exists $\alpha < 1$ such that $r + s + t = \alpha^2 rst$ or $\alpha(r + s + t) = \alpha^3 rst$. Let $a = \alpha r$, $b = \alpha s$, $c = \alpha t$. Write $a = \tan A$, $b = \tan B$, $c = \tan C$, then $A + B + C = \pi$. It is clear that

$$\begin{aligned}
\frac{1}{2} \times \text{LHS} &= \frac{1}{\sqrt{1+r^2}} + \frac{1}{\sqrt{1+s^2}} + \frac{1}{\sqrt{1+t^2}} \\
&< \frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \\
&= \cos A + \cos B + \cos B \\
&\leq 3 \cos\left(\frac{A+B+C}{3}\right) = \frac{3}{2} = \frac{1}{2} \times \text{RHS}.
\end{aligned}$$

**2nd soln:** Note that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{1}{xyz} \quad \Rightarrow \quad xy + yz + xz < 1.$$

Hence

$$\frac{2x}{\sqrt{1+x^2}} < \frac{2x}{\sqrt{x^2 + xy + xz + yz}} = \frac{2x}{\sqrt{(x+y)(x+z)}}.$$

By AM-GM we have

$$\frac{2x}{\sqrt{(x+y)(x+z)}} \leq \frac{x}{x+y} + \frac{x}{x+z}.$$

Similarly,

$$\frac{2y}{\sqrt{(y+z)(y+x)}} \leq \frac{y}{y+z} + \frac{y}{y+x}, \quad \frac{2z}{\sqrt{(z+x)(z+y)}} \leq \frac{z}{z+x} + \frac{z}{z+y}.$$

The desired inequality then follows by adding up the three inequalities.

**4.** Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$. Define $Q(x) = P(x+1) - P(x)$. Then $Q(x)$ is of degree $n - 1$. We'll prove by contradiction that $|Q(x)| \leq 3$ for all $x$. This will imply that $n \leq 1$. Assume that $|Q(a)| > 3$ for some $a \in \mathbb{R}$. Then $|P(a+1) - P(a)| > 3$. Thus there are 3 integers between $P(a)$ and $P(a+1)$. Hence there exists three values of $x \in [a, a+1]$ such that $P(x)$ is an integer. Thus there are three integers in $[a, a+1]$, a contradiction. There are two cases:

Case (i) $n = 0$. Here we have $P(x) = c$ where $c \notin \mathbb{Z}$.

Case (ii) $n = 1$. Here $P(x) = sx + t$. There are two integers $m, n$ such that $P(m) = sm + t = 0$ and $P(n) = sn + t = 1$. Thus $s(n - m) = 1$ implying that $1/s \in \mathbb{Z}$ and $sm + t = 0$ implying that $t/s \in \mathbb{Z}$. Letting $1/s = p$ and $t/s = q$, $P(x) = \frac{x}{p} + \frac{q}{p}$ where $p, q \in \mathbb{Z}$ and $p \neq 0$.

**5.** Answer: $(m, n) = (1144, 377)$ or $(377, 1144)$.

Let $m$ and $n$ be positive integers satisfying the given equation. That is $2011(m + n) = 3(m^2 - mn + n^2)$. Since the equation is symmetric in $m$ and $n$, we may assume $m \geq n$. If $m = n$, then $m = n = 4022/3$ which is not an integer. So we may further assume $m > n$. Let $p = m+n$ and $q = m-n > 0$. Then $m = (p+q)/2$ and $n = (p-q)/2$, and the equation becomes $8044p = 3(p^2 + 3q^2)$. Since 3 does not divide 8044, it must divide $p$. By letting $p = 3r$, the above equation reduces to $8044r = 3(3r^2 + q^2)$. From this, 3 must divide $r$. By letting $r = 3s$, we get $8044s = 27s^2 + q^2$, or equivalently

$$s(8044 - 27s) = q^2. \qquad (*)$$

For $s$ between 1 and $\lfloor 8044/27 \rfloor = 297$, the number $s(8044 - 27s)$ is a square only when $s = 169$. To narrow down the values of $s$, we proceed as follow.

Let $s = 2^\alpha u$, where $\alpha$ is a nonnegative integer and $u$ is an odd positive integer. Suppose $\alpha$ is odd and $\alpha \geq 3$. Then $(*)$ becomes $2^{\alpha+2}u(2011 - 27 \times 2^{\alpha-2}u) = q^2$ which is a square. Since $\alpha + 2$ is odd, 2 must divide $2011 - 27 \times 2^{\alpha-2}u$ implying 2 divides 2011 which is a contradiction. Next suppose $\alpha = 1$. Then we have $u(2 \times 2011 - 27u) = (q/2)^2$. If $u$ is not a square, then there exists an odd prime factor $t$ of $u$ such that $t$ divides $2 \times 2011 - 27u$. Thus $t$ divides $2 \times 2011$ so that $t$ must be 2011 since 2011 is a prime. But then $u \geq t = 2011$ contradicting $2 \times 2011 - 27u > 0$. Therefore $u$ must be a square. This implies that $2 \times 2011 - 27u$ is also a square. Taking mod 4, we have $u \equiv 0$ or $1 \pmod 4$ so that $2 \times 2011 - 27u \equiv 2$ or $3 \pmod 4$ which contradicts the fact that $2 \times 2011 - 27u$ is a square. Thus $\alpha \neq 1$ too. Consequently $\alpha$ must be even. Then dividing both sides of $(*)$ by $2^\alpha$, we obtain $u(8044 - 27 \times 2^\alpha u) = q^2/2^\alpha$ which is a square. Now suppose $u$ is not a square. Then there exists an odd prime factor $v$ of $u$ such that $v$ divides $8044 - 27 \times 2^\alpha u$. Then $v$ must divide 8044 so that $v = 2011$. Thus $u \geq v = 2011$. This again contradicts the fact that $8044 - 27 \times 2^\alpha u > 0$. Therefore $u$ is a square. Consequently $s$ is also a square. Write $s = w^2$. Then $(*)$ becomes $w^2(8044 - 27w^2) = q^2 \geq 0$. From this $w \leq \lfloor (8044/27)^{\frac{1}{2}} \rfloor = 17$. A direct verification shows that $8044 - 27w^2$ is a square only when $w = 13$. Thus $s = w^2 = 169$. Then $p = 3r = 9s = 1521$, and by $(*)$ $q = (169 \times (8044 - 27 \times 169))^{\frac{1}{2}} = 767$. Lastly, $m = (p+q)/2 = 1144$ and $n = (p-q)/2 = 377$.